SOME THOUGHTS ABOUT THE MONTE CARLO METHOD WITH STRATIFIED SAMPLING

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OVERVIEW

- ✓ Analysis of output pdf's with Monte Carlo
- ✓ Re-arranging arrays
- ✓ Stratified Sampling
- ✓ Examples of Stratified Sampling for Gaussians, Student's-t and U-shaped distributions
- ✓ How to reproduce coverage factors (from the GUM) related to the combination of two contribuions, and a look at the associated pdf's
- ✓ Example of application: Mismatch correction in characterisation of power sensors (complex formulas involved)
- ✓ Further methods to simplify Monte Carlo: estimating bounds for the combination of complex quantities

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HOW DO WE COPE WITH THE OUTPUT PDF USING MONTE CARLO?

- ✓ One of the advantages of the Monte Carlo method is that we can apply the usual statistical tools, in order to determine Mean Value, the confidence interval, the uncertainty associated to a level of confidence of 95%, the possible existence of asymmetrical intervals, etc.

- ✓ To do this, the user just needs to ‘count’ generated points. In other words, my spreadsheet orders the data and sees in which interval a given percentage of occurrences is found, such as the already mentioned 95% of generated data. No mathematical or statistical skills are required to analyse the probability density functions (pdf’s) obtained at the output.

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WHAT IF THE OUTPUT PDF IS REALLY ODD?

- ✓ In occasions we find it difficult when an output pdf is obtained, as a result of a Monte Carlo computation, which greatly differs from the usual Gaussian of Student's-t distributions.
- ✓ An arbitrary pdf can be obtained, even asymmetrical, and this is difficult to describe in terms of uncertainty intervals related to a confidence level of 95%, as we are used to do with the most popular pdf's which are included in the calibration certificates.
- ✓ However, in a world in which we all were working with the Monte Carlo method, perhaps it would not be necessary to reduce the output pdf's obtained in our simulations to such simple parameters as a Mean Value and an expanded uncertainty interval.
- ✓ As a result of my simulation, I could be arriving at a predicted pdf described in terms of a 'cloud' or set of simulated points, this being the starting point of subsequent computations.
- ✓ This can be the case in the characterization of a working standard to be included in subsequent computations when using this specific DUT. Or when the DUT belongs to a customer and I provide him or her with the simulated 'cloud', for his or her records and to be used in computations further ahead in the traceability chain. This would be the equivalent to providing a calibration certificate.

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HOW TO MAKE USE OF THE OUTPUT PDF AS OBTAINED WITH MONTE CARLO?

- ✓ The output pdf, as obtained with Monte Carlo and described by a set of values (the ‘cloud’) following a probability density function which may be of arbitrary shape, is in turn the input pdf to subsequent computations made in my laboratory or by my customer. So let us provide the clients with the ‘cloud’ of points in electronic format.

- ✓ This reproduces the idea of the calibration certificate, providing the traceability and the metrological knowledge of the device under test. The difference is that in a certificate we only include some (few) parameters describing in a convenient way the obtained pdf (either Gaussian or Student’s-t) in terms of Mean Value, expanded uncertainty, the coverage factor used, the effective number of degrees of freedom, the agreed confidence level of 95%, etc.

- ✓ We are all familiar with these parameters and have not even thought of the possibility of obtaining pdf’s of different shapes (how about a combination of Gaussian and U-shaped, for example?)

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IN A SET OF GENERATED DATA, WHERE CAN WE FIND RANDOMNESS?

- ✓ It is true that the 'cloud' of points which I am providing my customer with (or which I am receiving from a laboratory at a higher level in the traceability chain) will always be the same, there is nothing like randomness in it.
- ✓ The proposed solution: let us arrange it in a different order!
- ✓ Admitting that a set of generated data is everything but a random phenomenon, and bearing in mind that we are dealing with the Monte Carlo method… we are missing something, right?
- ✓ Whenever I run a Monte Carlo simulation which makes use of this set of previously generated data (now as an input pdf describing my travelling standard) I can randomly re-arrange (or 'shuffle') the original set of points for subsequent computations.

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RE-ARRANGING DATA IS A GOOD IDEA!

- ✓ And there are pre-defined functions such as **randperm** in Mathlab® which allows us to randomly permute an array. In Excel® the function **RANK** can be used to order a set of random numbers and to extract the position that each number occupies in the original array.
- ✓ Re-arranging arrays is the key point to the following concept in Monte Carlo: Stratified Sampling.

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STRATIFIED SAMPLING

- ✓ First goal of the Stratified Sampling of data sets are: it may save up computational time or reduce the size of the required Excel® files. Or more practical, it can produce 'beautiful', near theoretical histograms – or pdf's.

- ✓ Let us begin with a uniform histogram between 0 and 1, defined by these ten values: 0.05 – 0.15 – 0.25 – 0.35 – 0.45 – 0.55 – 0.65 – 0.75 – 0.85 – 0.95. Once generated, the original array can be re-arranged, obtaining a perfect histrogramm for ten values, with ten categories and a frequency of 1 for each category. Every time Monte Carlo is run, the same histogram is obtained, but with a different permutation.

- ✓ We can use 100 values: 0.005 – 0.015 – […] – 0.985 – 0.995. Or 1000 values: 0.0005 – 0.0015 – […] – 0.9985 – 0.9995. Working wiht a fixed number of categories, let us say 20, we would be obtaining histrogramms with frequencies 5 or 50, respectively. Always perfect histrogramms.

- ✓ General formula for N points: 1/2/N – […] – (2N-1)/2/N in 1/N steps. In a histrogramm with m categories, the frequency obtained is N/m.

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SOME HINTS TO 'SHUFFLE' ARRAYS

✓ With Matlab® it is straightforward to define an array containing the above points:

linspace(1/2/N, (2*N-1)/2/N, N)

✓ And then to 'shuffle' it:

randperm(**linspace**(1/2/N, (2*N-1)/2/N, N))

✓ With Excel® the process requires a little bit more space: to define a column between rows 1 and N containing the original stratified array, use can be made of the predefined function **ROW**:

ROW()/N-1/2/N

✓ But the index represented by **ROW** is ordered. To 'shuffle' the array, let us 'shuffle' the index as follows: we define an auxiliary **range** containing pure random numbers between 0 and 1, generated through the Excel® function **RANDOM**. Then we substitute **ROW** by **RANK** in the above expression. For the first cell we point at the first cell in the auxiliary array:

RANK(1stCell;range)/N-1/2N

✓ And finally we extend the formula to the rest of cells in the desired column (note that we need two columns for each array required).

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SO FAR IT SEEMS TO BE SIMPLE BUT USELESS…

- ✓ Not at all if we wish to simulate a Gaussian, a Student's-t or a U-shaped distribution. In all cases we make use of the stratified rectangular distribution as input to to generate the desired distribution as output. And we make this by means of the inverse CDF (Cumulative Distribution Function).

- ✓ In the case of a Gaussian distribution, with Mean Value 0 and Standard Deviation 1, let us try with this expression, which points out to the array contained in **range** (**range** contains the stratified rectangular distribution between 0 and 1):

NORMINV(range;0;1)

- ✓ If we wish to generate a Student's-t distribution with Dof degrees of freedom:

IF(range<0.5;-**TINV**(2*range;Dof);**TINV**(2*(1-range);Dof))

- ✓ Or in the case of a U-shaped distribution between -1 and 1, just calling the sine function:

SIN(**PI**()*range-**PI**()/2)

- ✓ In all examples, the histograms obtained are 'perfect', or at least as perfect as the density of the stratified sampling of the rectangular distribution allows. The histograms become more populated and more 'beautiful' as the number of points is increased, but the result is always the same, not dependant on randomness.

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THE TAILS OF THE GAUSSIANS ARE A FIGURE OF MERIT IN SIMULATIONS

- ✓ Key point in the simulation of Gaussian distributions are the obtained tails. With a pure random method, one is never sure to be obtaining the desired tails which assure a given precision of the simulation (the greater the ratio ‘extension of tail vs. sigma’ the better the simulation or the more sure one is about having included the ‘probability of extrem occurences’ in the generated set of data.

- ✓ With Stratified Sampling, there is a mathematical relationship between the number of points N and the extensión of tails (maximum and minimum data of the generated array divided by the standard deviation, which ideally tends to plus / minus infinity when the simulation is made finer and finer).

- ✓ The ratio ‘tails vs. sigma’ gives us an idea obout the quality of the simulation. If one is able to include tails of $\pm 4 \cdot \sigma$ representing 99.99% of the theoretical Gaussian, the simulation will be better than another one just including $\pm 3 \cdot \sigma$ or 99.73% of possible occurrences.

- ✓ With Stratified Sampling we can tabulate the necessary number of points N in order to get a given extension of tails: the ratio ‘tails vs. Sigma’ is given by **NORMINV**(1/2N,0,1). Whenever I make use of a simulation with N points, the same histogram is obtained, with constant tails ranging from **NORMINV**(1/2N,0,sigma) to **NORMINV**(1/2N,0,sigma). So I can choose N as a function of the requirements of my simulation.

- ✓ The tails can be also be computed and tabulated for a Student’s-t distribution with a given number of degrees of freedom. Or for a U-shaped distribution.

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EXAMPLE 1 – SIMULATION OF GAUSSIANS WITH STRATIFIED SAMPLING

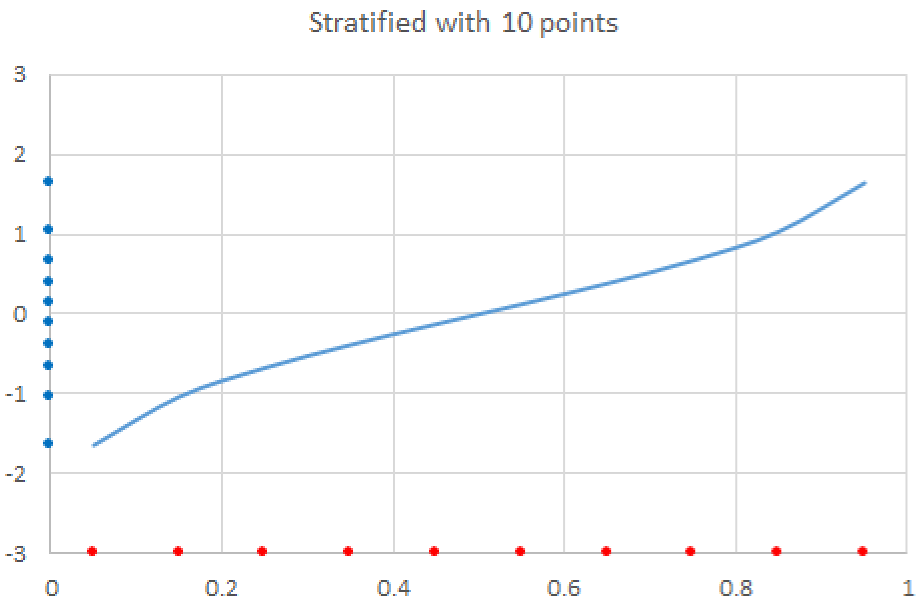

Fig. 1. Gaussian distribution, Stratified Sampling with N=10 points
Tails: -1.6449 to +1.6449

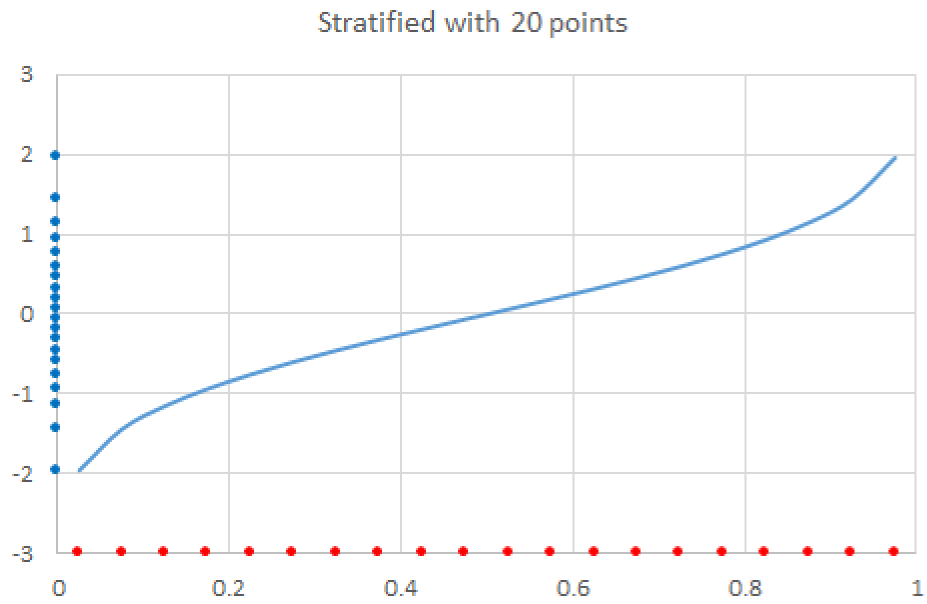

Fig. 2. Gaussian distribution, Stratified Sampling with N=20 points
Tails: -1.9600 to +1.9600

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EXAMPLE 1 – SIMULATION OF GAUSSIANS WITH STRATIFIED SAMPLING

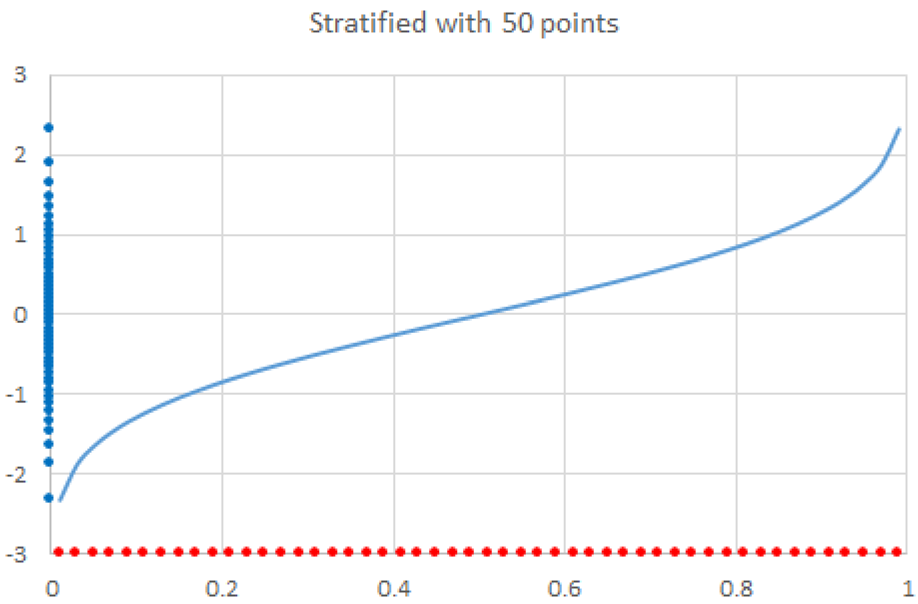

Fig. 3. Gaussian distribution, Stratified Sampling with N=50 points
Tails: -2.3263 to +2.3263

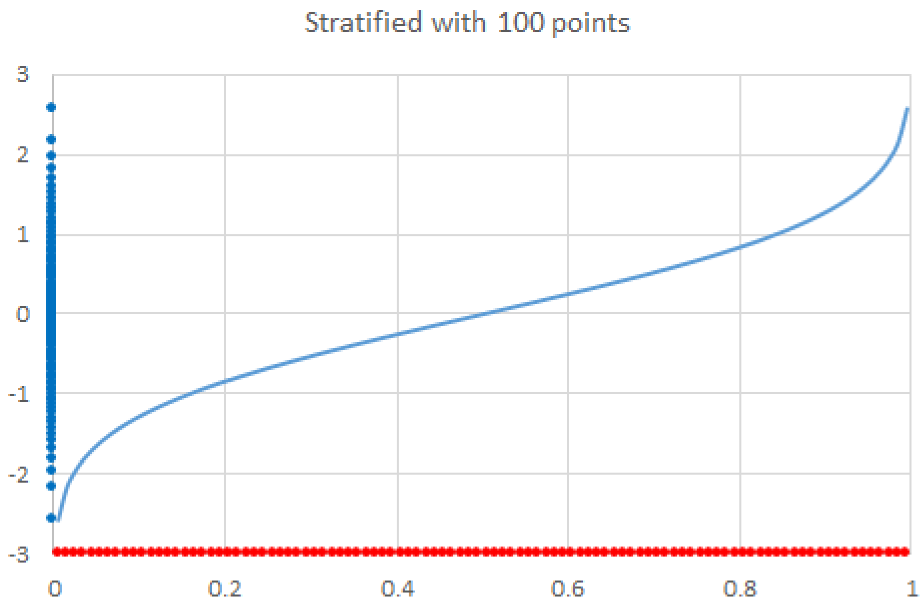

Fig. 4. Gaussian distribution, Stratified Sampling with N=100 points
Tails: -2.5758 to +2.5758

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EXAMPLE 2 – SIMULATION OF U-SHAPED WITH STRATIFIED SAMPLING

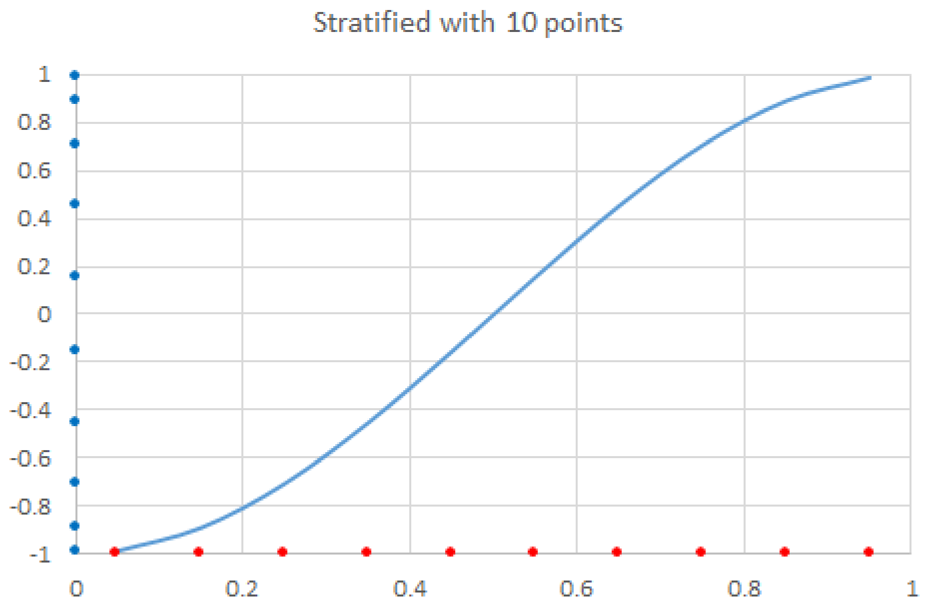

Fig. 5. U-shaped distribution, Stratified Sampling with N=10 points
Tails: -0.9877 to +0.9877

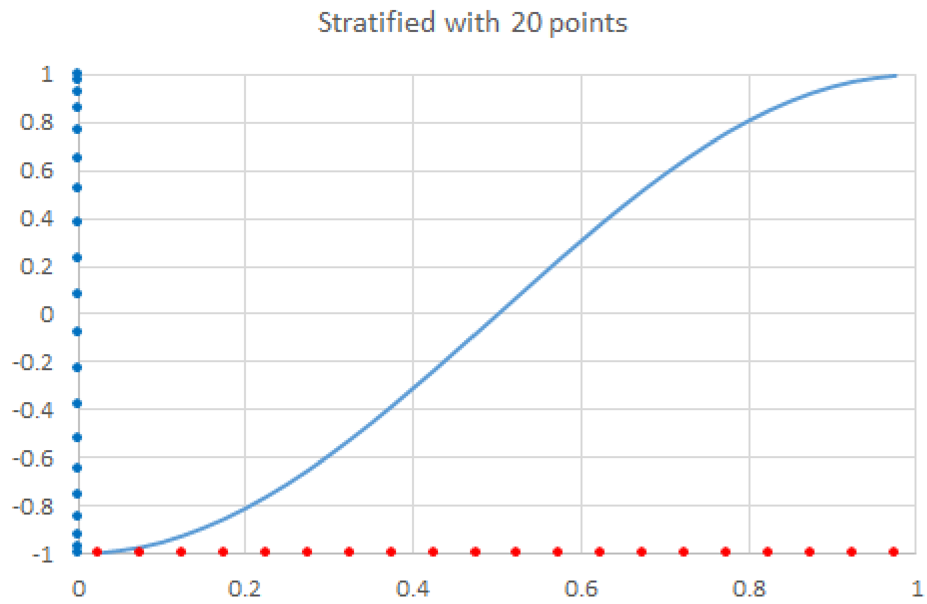

Fig. 6. U-shaped distribution, Stratified Sampling with N=20 points
Tails: -0.9969 to +0.9969

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EXAMPLE 2 – SIMULATION OF U-SHAPED WITH STRATIFIED SAMPLING

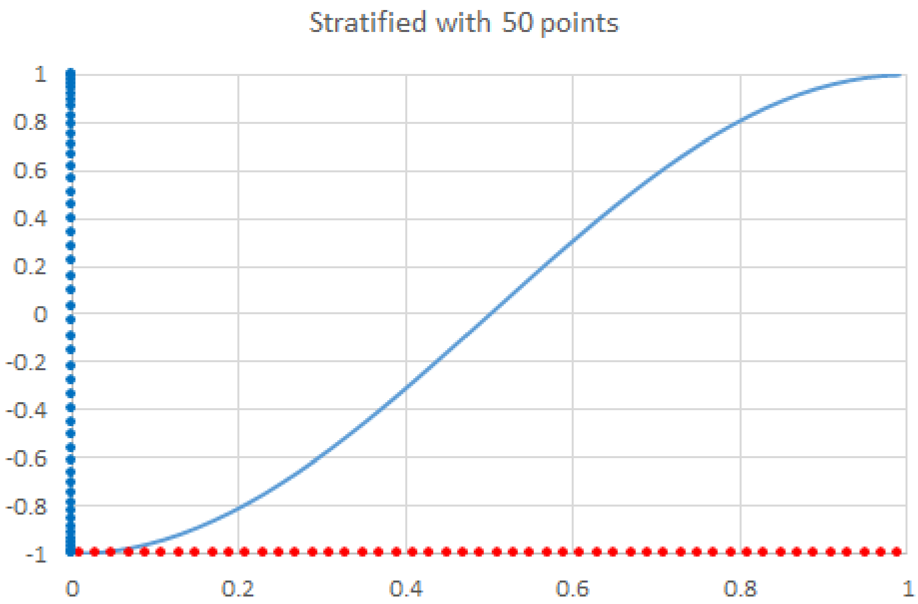

Fig. 7. U-shaped distribution, Stratified Sampling with N=50 points
Tails: -0.9995 to +0.9995

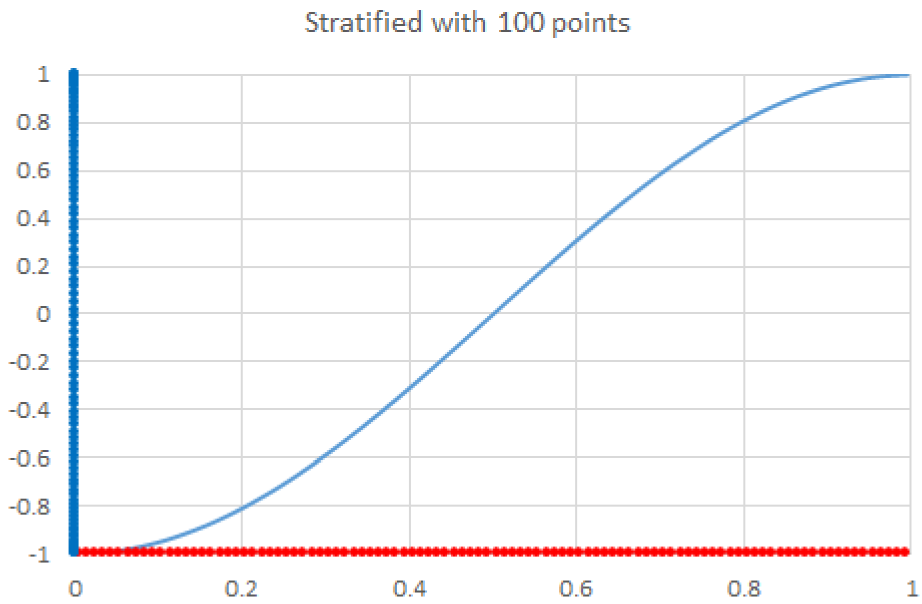

Fig. 8. U-shaped distribution, Stratified Sampling with N=100 points
Tails: -0.9999 to +0.9999

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EXAMPLE 3 – SIMULATION OF STUDENT'S-T WITH STRATIFIED SAMPLING

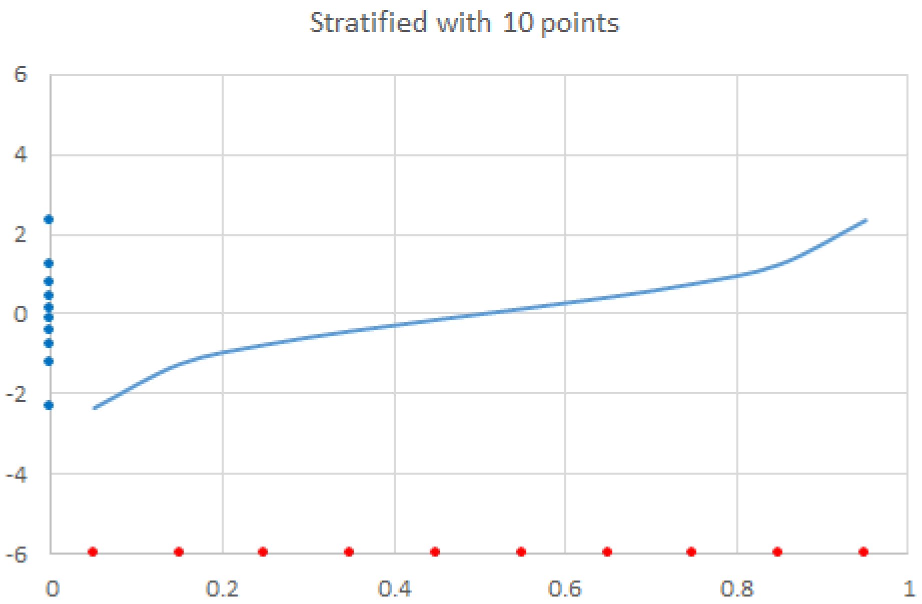

Fig. 9. Student's-t distribution, Dof=3, Stratified Sampling with N=10 points
Tails: -2.3534 to +2.3534

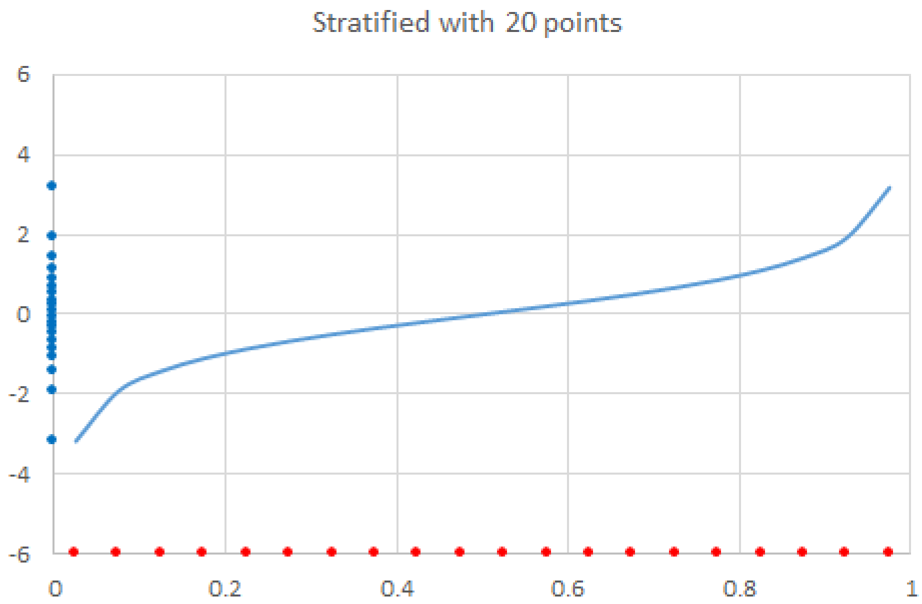

Fig. 10. Student's-t distribution, Dof=3, Stratified Sampling with N=20 points
Tails: -3.1824 to +3.1824

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EXAMPLE 3 – SIMULATION OF STUDENT'S-T WITH STRATIFIED SAMPLING

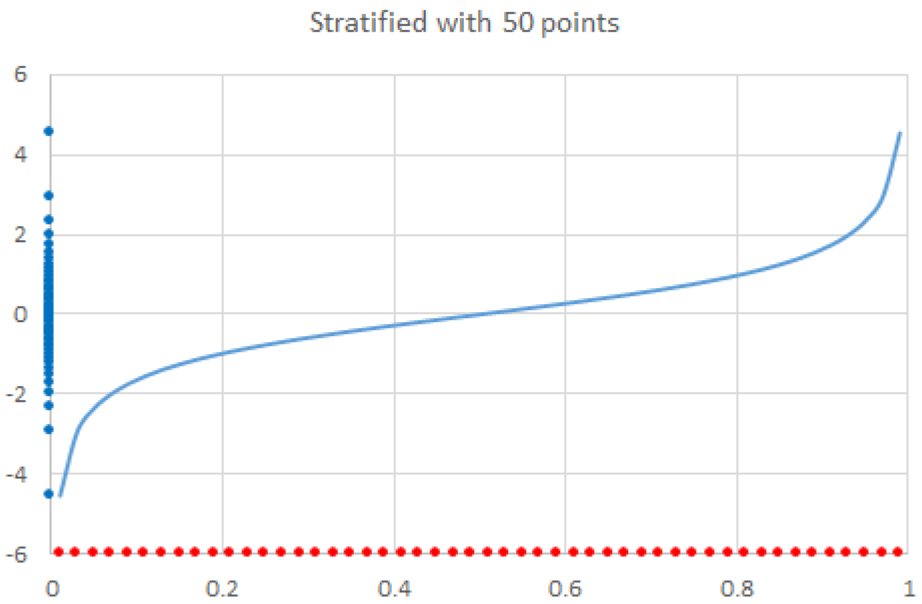

Fig. 11. Student's-t distribution, Dof=3, Stratified Sampling with N=50 points
Tails: -4.5407 to +4.5407

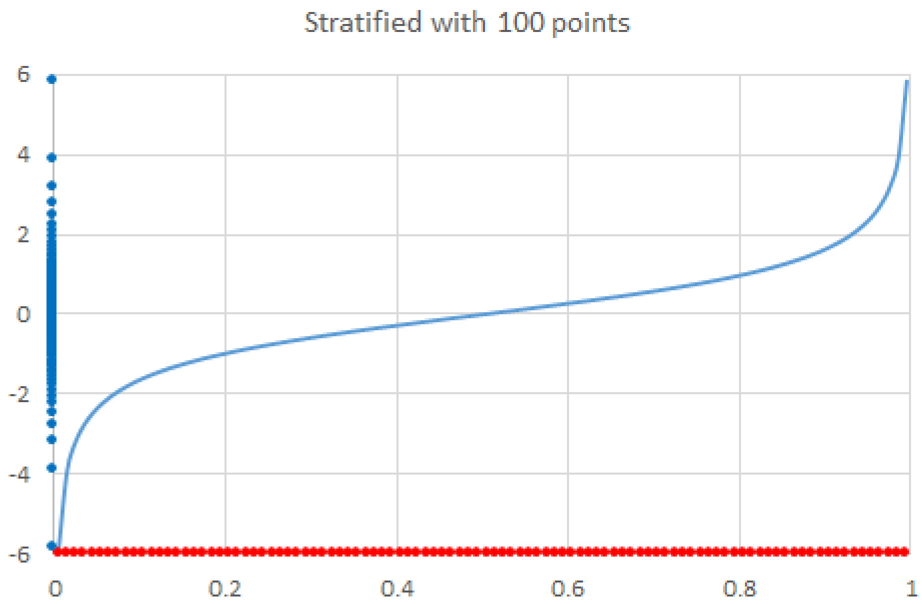

Fig. 12. Student's-t distribution, Dof=3, Stratified Sampling with N=100 points
Tails: -5.8409 to +5.8409

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COMBINATION OF A GAUSSIAN AND A RECTANGULAR DISTRIBUTIONS – THEORETICAL

The following table has been obtained from "The Expression of Uncertainty and Confidence in Measurement", M3003, Edition 3, November 2002:

$\frac{u_i(y)_{normal}}{u_i(y)_{rect}}$	$k_{95.45}$	$\frac{u_i(y)_{normal}}{u_i(y)_{rect}}$	$k_{95.45}$	$\frac{u_i(y)_{normal}}{u_i(y)_{rect}}$	$k_{95.45}$
0.00	1.65	0.50	1.84	0.95	1.95
0.10	1.66	0.55	1.85	1.00	1.95
0.15	1.68	0.60	1.87	1.10	1.96
0.20	1.70	0.65	1.89	1.20	1.97
0.25	1.72	0.70	1.90	1.40	1.98
0.30	1.75	0.75	1.91	1.80	1.99
0.35	1.77	0.80	1.92	2.00	1.99
0.40	1.79	0.85	1.93	2.50	2.00
0.45	1.82	0.90	1.94	∞	2.00

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COMBINATION OF A GAUSSIAN AND A RECTANGULAR DISTRIBUTIONS – SIMULATED

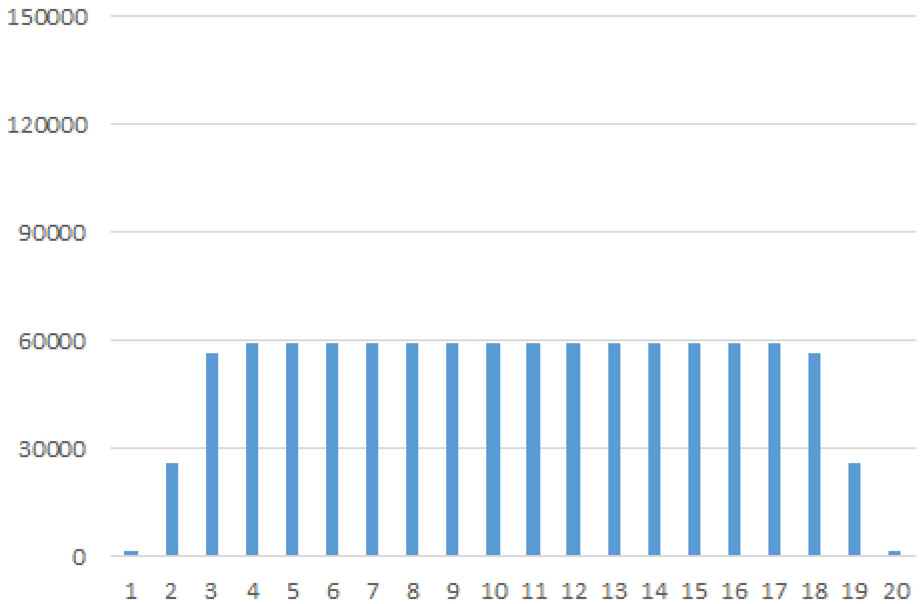

Fig. 13. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.10
$k_{95.45}$ = 1.66

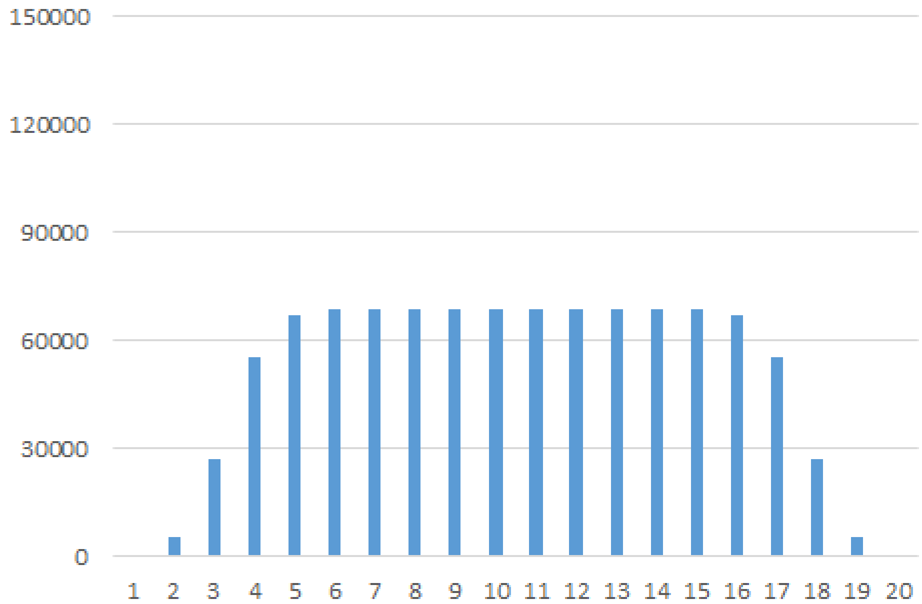

Fig. 14. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.20
$k_{95.45}$ = 1.70

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COMBINATION OF A GAUSSIAN AND A RECTANGULAR DISTRIBUTIONS – SIMULATED

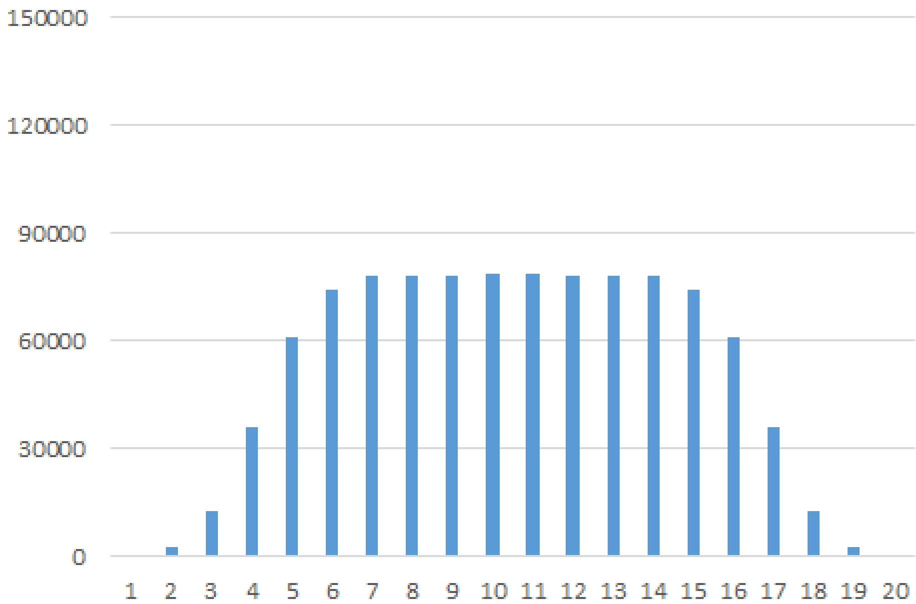

Fig. 15. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.30
$k_{95.45}$ = 1.75

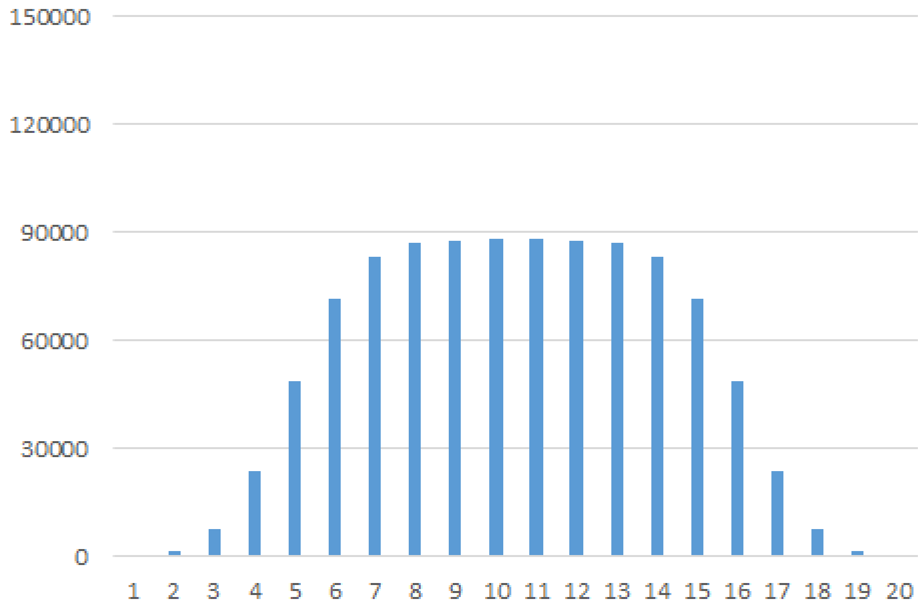

Fig. 16. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.40
$k_{95.45}$ = 1.79

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COMBINATION OF A GAUSSIAN AND A RECTANGULAR DISTRIBUTIONS – SIMULATED

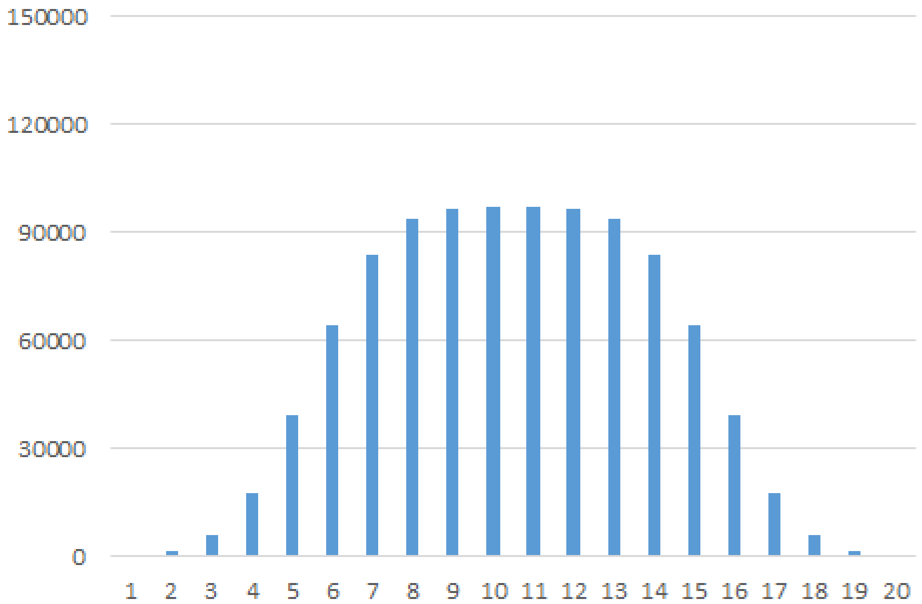

Fig. 17. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.50
$k_{95.45} = 1.84$

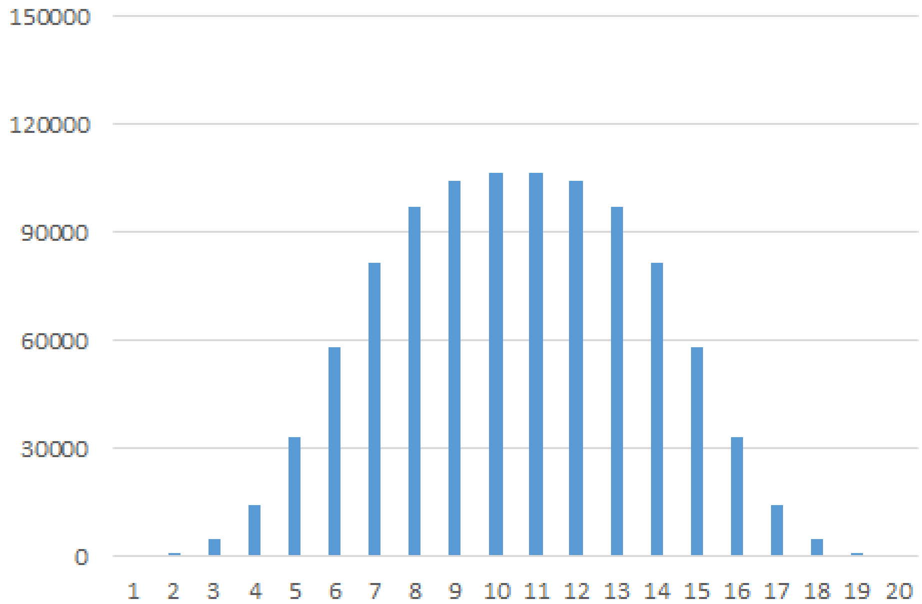

Fig. 18. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.60
$k_{95.45} = 1.87$

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COMBINATION OF A GAUSSIAN AND A RECTANGULAR DISTRIBUTIONS – SIMULATED

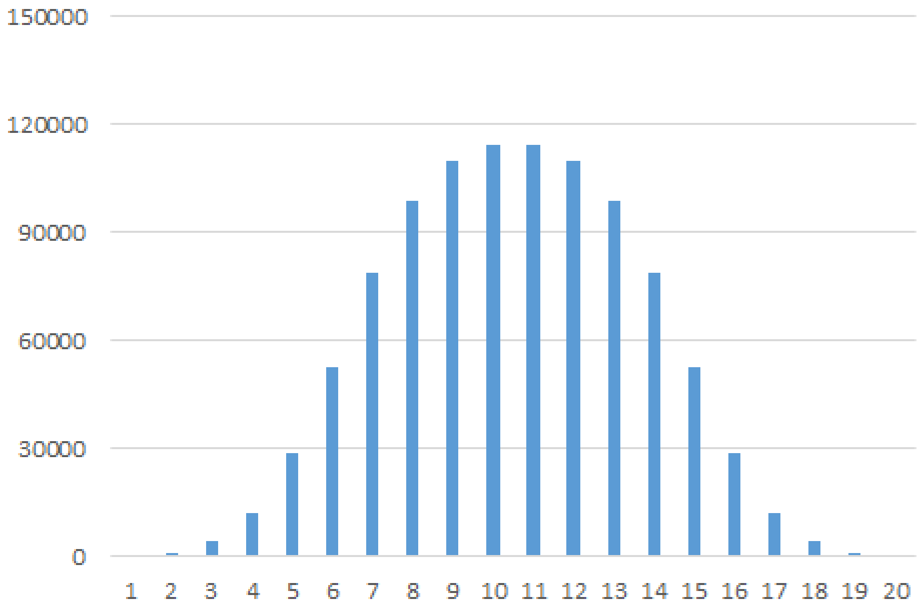

Fig. 19. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.70
$k_{95.45}$ = 1.90

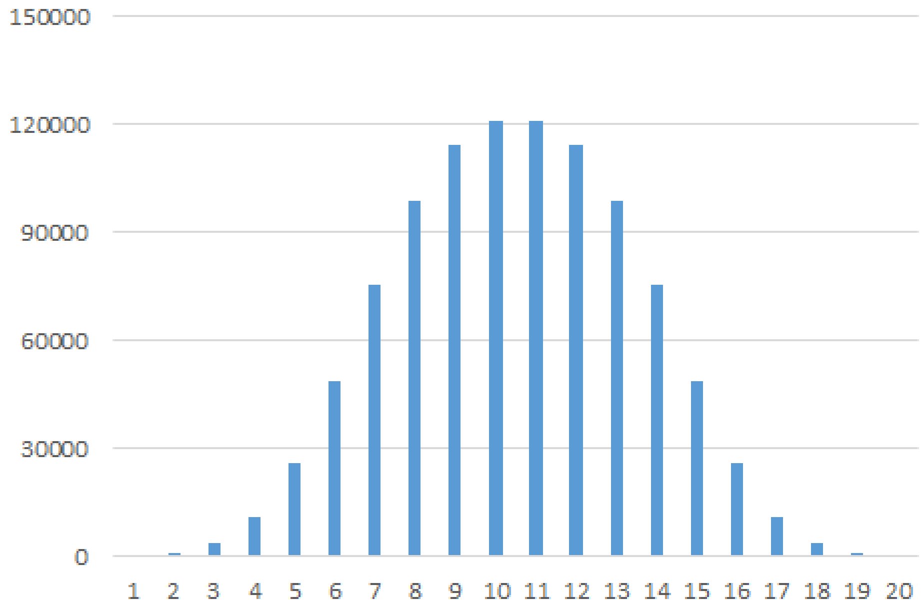

Fig. 20. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.80
$k_{95.45}$ = 1.92

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COMBINATION OF A GAUSSIAN AND A RECTANGULAR DISTRIBUTIONS – SIMULATED

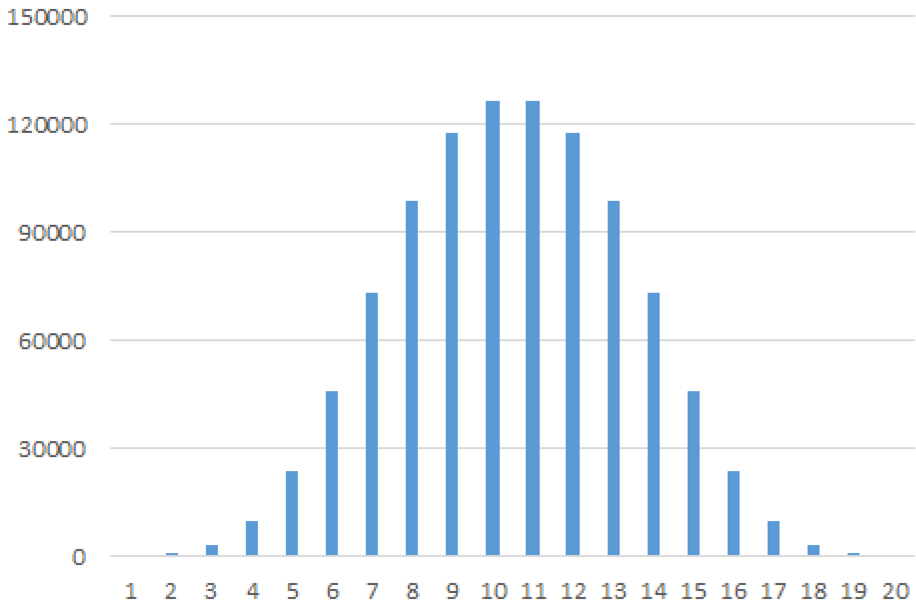

Fig. 21. $u_i(y)$ normal / $u_i(y)$ rectangular = 0.90
$k_{95.45}$ = 1.94

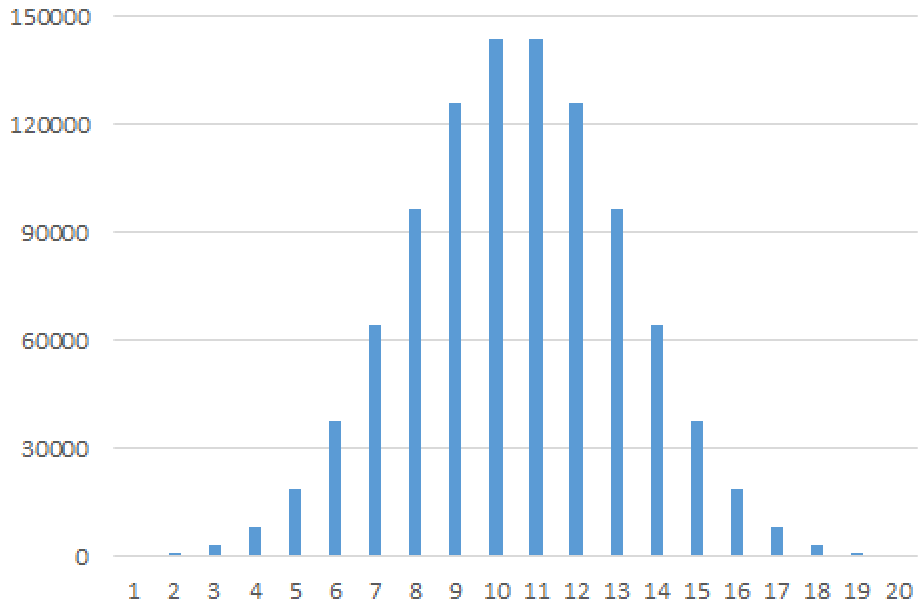

Fig. 22. $u_i(y)$ normal / $u_i(y)$ rectangular = 2.00
$k_{95.45}$ = 1.99

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COMBINATION OF A GAUSSIAN AND A U-SHAPED DISTRIBUTIONS – THEORETICAL

The following table has been obtained from “The Expression of Uncertainty and Confidence in Measurement”, M3003, Edition 3, November 2002:

$\frac{u_i(y)_{normal}}{u_i(y)_{U-shaped}}$	$k_{95.45}$	$\frac{u_i(y)_{normal}}{u_i(y)_{U-shaped}}$	$k_{95.45}$	$\frac{u_i(y)_{normal}}{u_i(y)_{U-shaped}}$	$k_{95.45}$
0.00	1.41	0.50	1.77	0.95	1.93
0.10	1.47	0.55	1.80	1.00	1.93
0.15	1.51	0.60	1.82	1.10	1.95
0.20	1.55	0.65	1.84	1.20	1.96
0.25	1.60	0.70	1.86	1.40	1.97
0.30	1.64	0.75	1.88	1.80	1.99
0.35	1.67	0.80	1.89	2.00	1.99
0.40	1.71	0.85	1.90	2.50	2.00
0.45	1.74	0.90	1.92	∞	2.00

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COMBINATION OF A GAUSSIAN AND A U-SHAPED DISTRIBUTIONS – SIMULATED

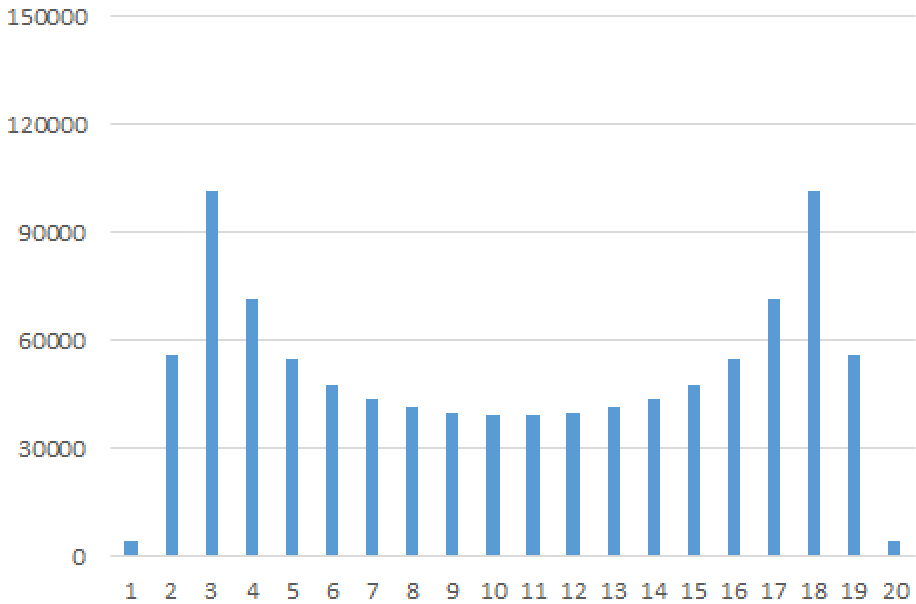

Fig. 23. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.10
$k_{95.45} = 1.47$

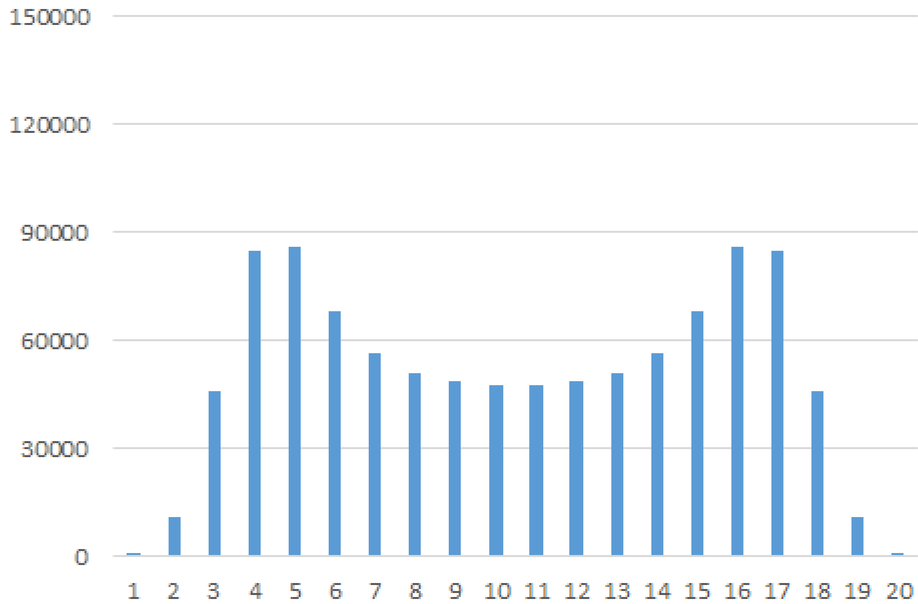

Fig. 24. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.20
$k_{95.45} = 1.55$

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COMBINATION OF A GAUSSIAN AND A U-SHAPED DISTRIBUTIONS – SIMULATED

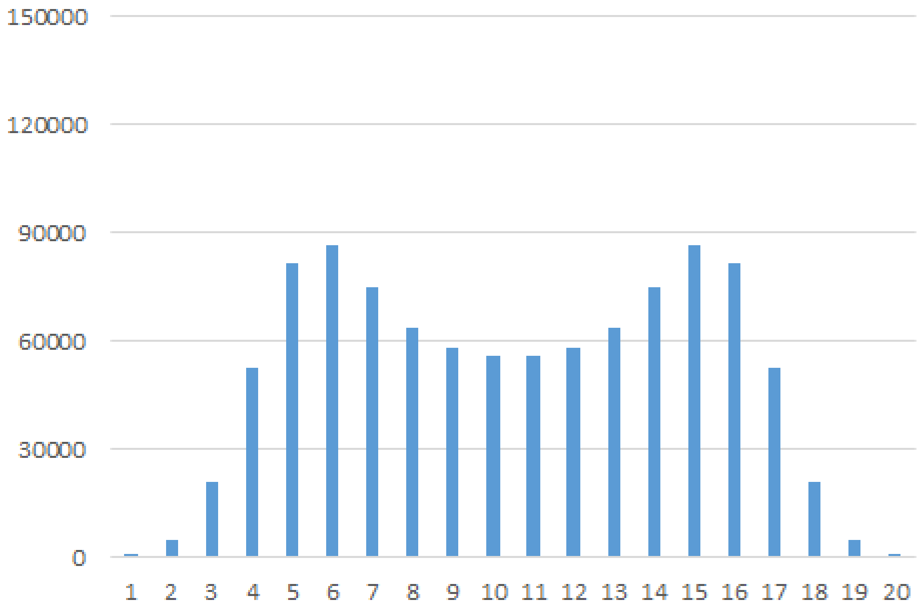

Fig. 25. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.30
$k_{95.45} = 1.64$

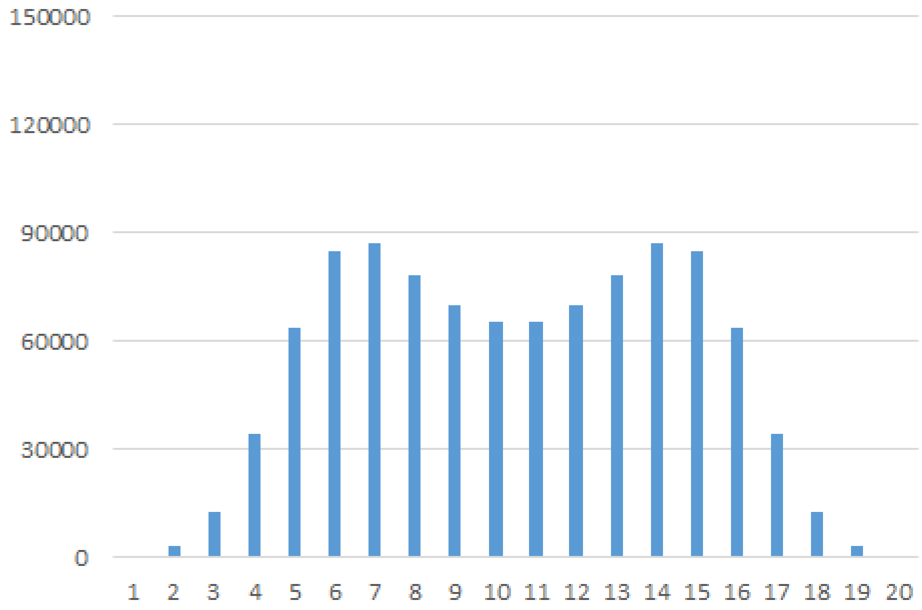

Fig. 26. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.40
$k_{95.45} = 1.71$

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COMBINATION OF A GAUSSIAN AND A U-SHAPED DISTRIBUTIONS – SIMULATED

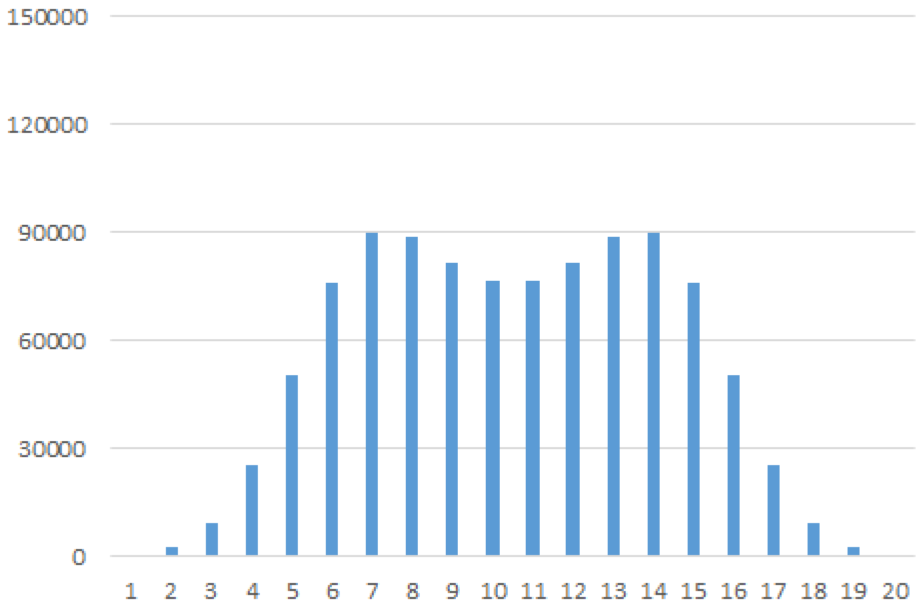

Fig. 27. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.50
$k_{95.45}$ = 1.77

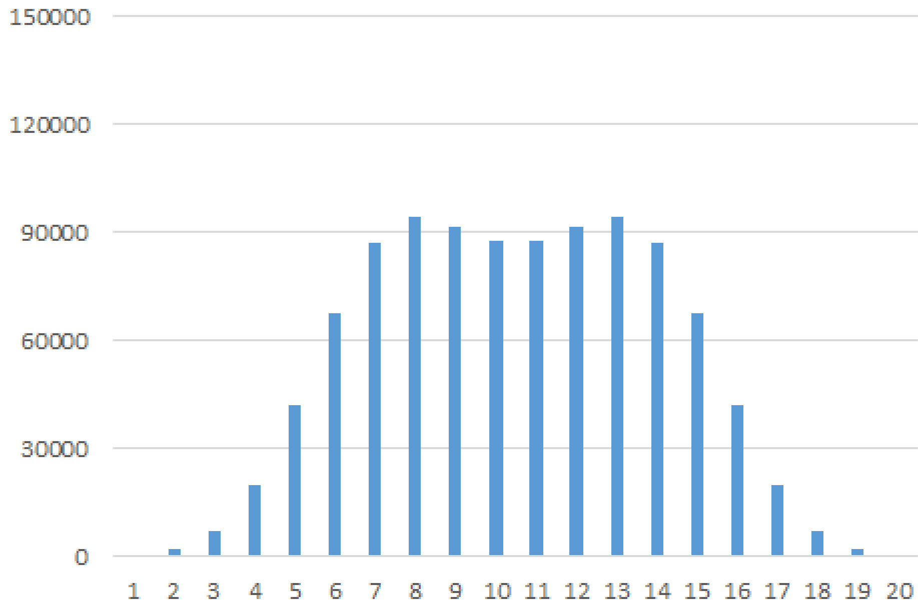

Fig. 28. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.60
$k_{95.45}$ = 1.82

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COMBINATION OF A GAUSSIAN AND A U-SHAPED DISTRIBUTIONS – SIMULATED

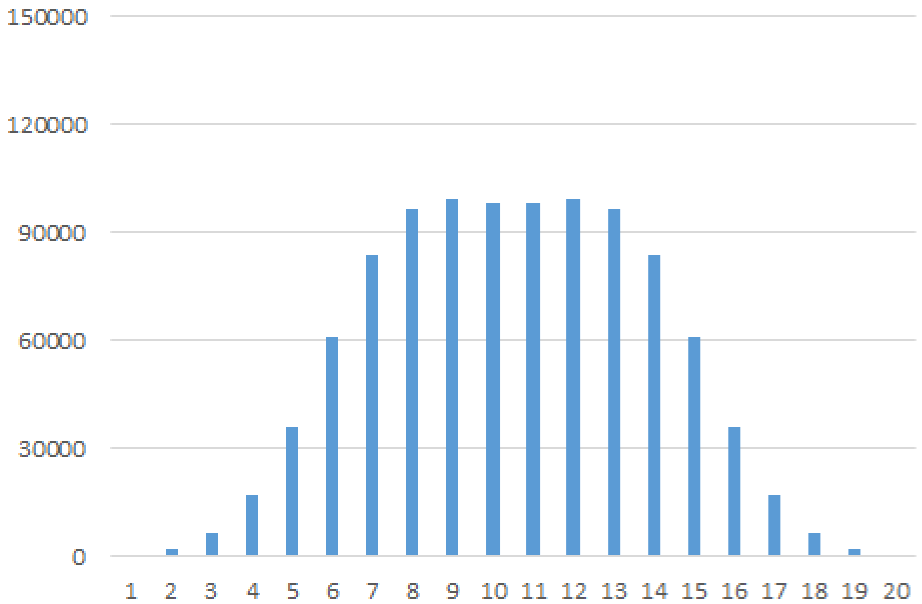

Fig. 29. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.70
$k_{95.45}$ = 1.86

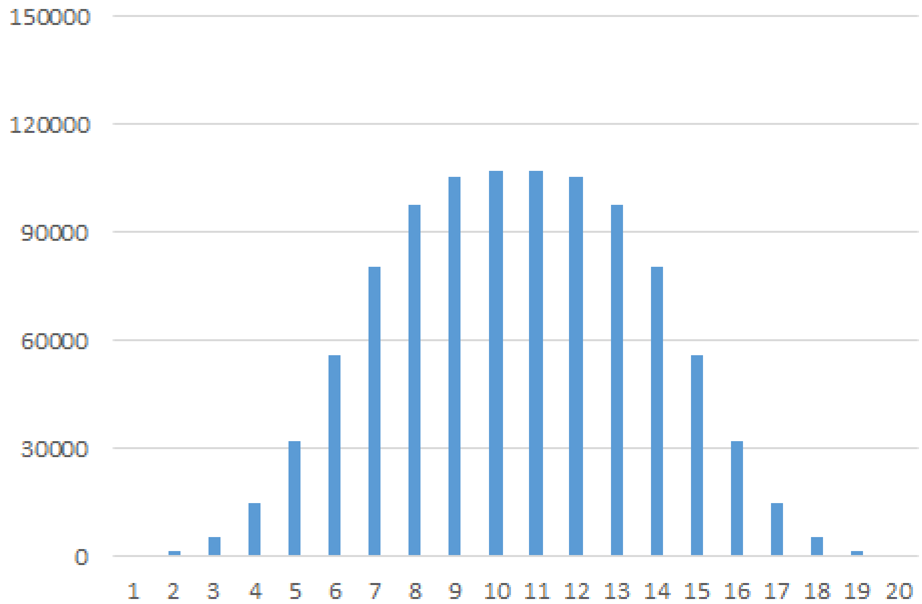

Fig. 30. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.80
$k_{95.45}$ = 1.89

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COMBINATION OF A GAUSSIAN AND A U-SHAPED DISTRIBUTIONS – SIMULATED

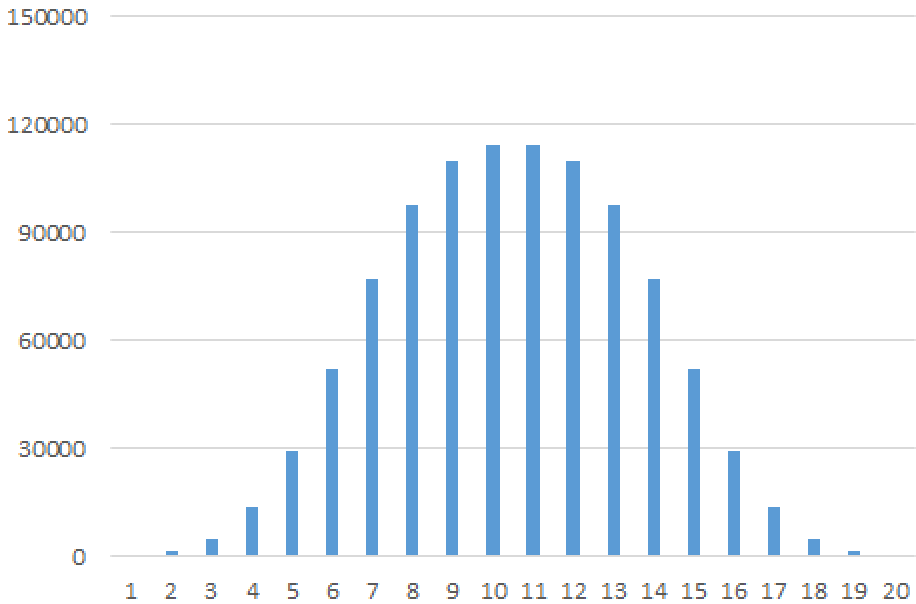

Fig. 31. $u_i(y)$ normal / $u_i(y)$ U-shaped = 0.90
$k_{95.45}$ = 1.92

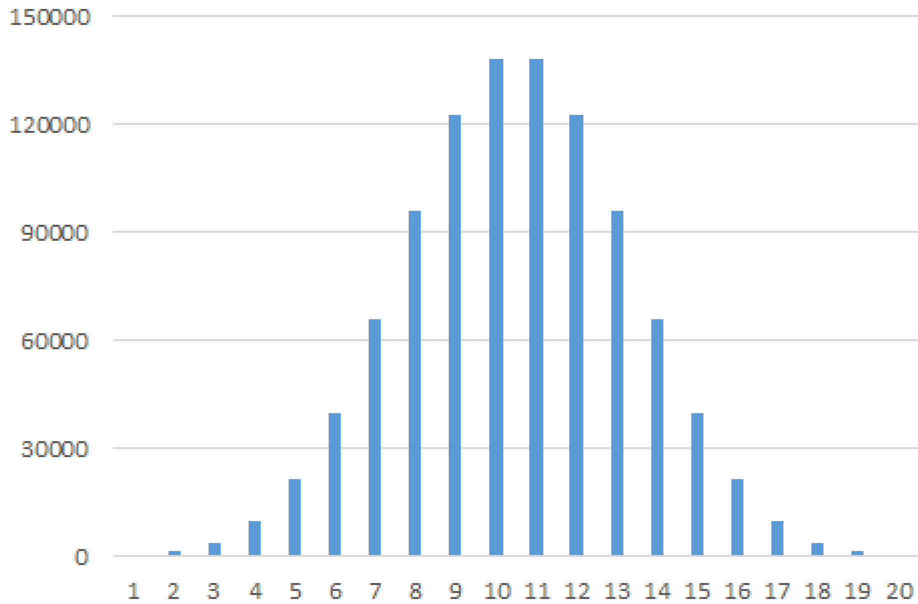

Fig. 32. $u_i(y)$ normal / $u_i(y)$ U-shaped = 2.00
$k_{95.45}$ = 1.99

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COMBINATION OF A RECTANGULAR AND A U-SHAPED DISTRIBUTIONS – THEORETICAL

The following table has been obtained from “The Expression of Uncertainty and Confidence in Measurement”, M3003, Edition 3, November 2002:

$\frac{u_i(y)_{rect}}{u_i(y)_{U-shaped}}$	$k_{95.45}$	$\frac{u_i(y)_{rect}}{u_i(y)_{U-shaped}}$	$k_{95.45}$	$\frac{u_i(y)_{rect}}{u_i(y)_{U-shaped}}$	$k_{95.45}$
0.00	1.41	0.45	1.75	3.0	1.80
0.10	1.48	0.50	1.78	4.0	1.75
0.15	1.53	0.60	1.82	5.0	1.72
0.20	1.57	0.70	1.86	6.0	1.70
0.25	1.62	0.80	1.88	7.5	1.68
0.30	1.66	0.90	1.89	10	1.66
0.35	1.69	1.0	1.90	20	1.65
0.40	1.73	2.0	1.86	∞	1.65

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COMBINATION OF A RECTANGULAR AND A U-SHAPED DISTRIBUTIONS – SIMULATED

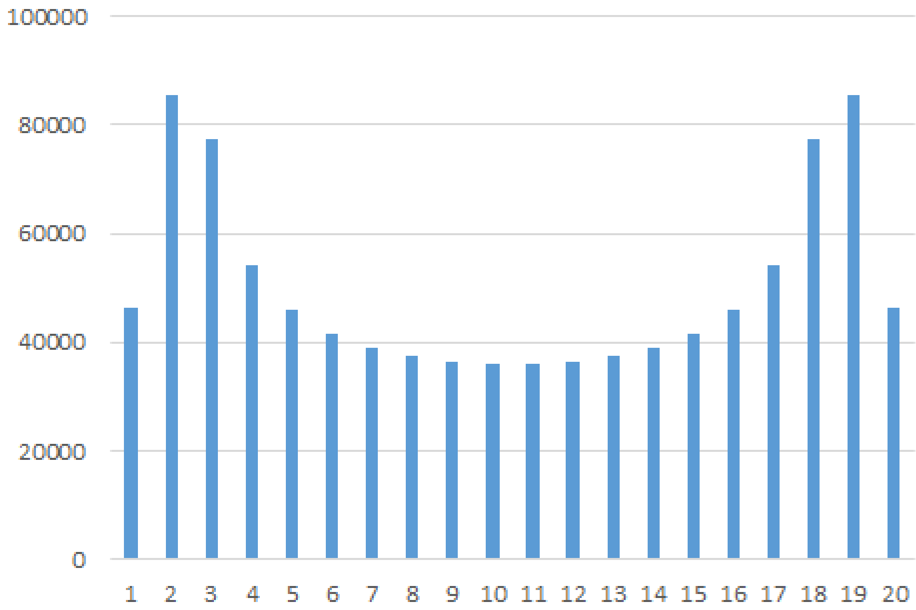

Fig. 33. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 0.10
$k_{95.45} = 1.48$

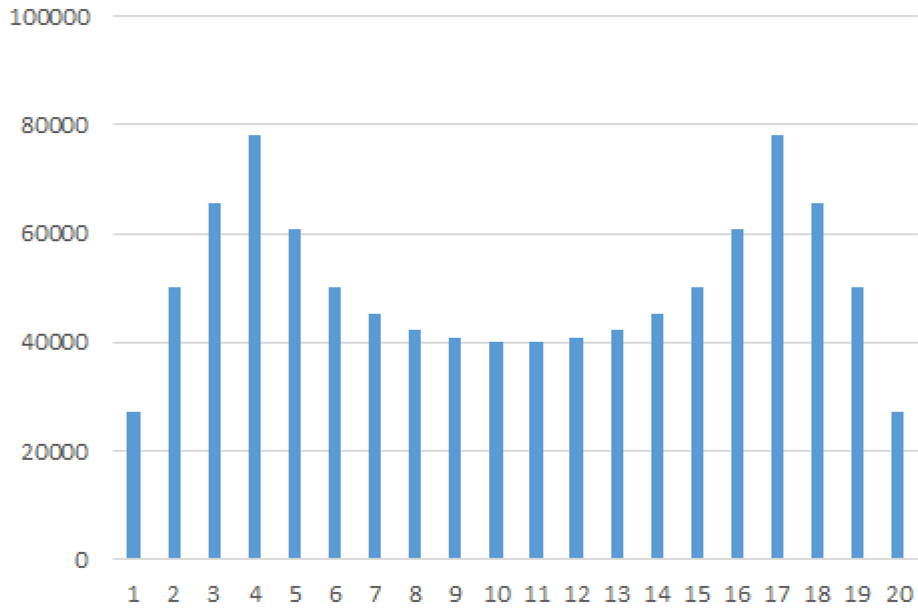

Fig. 34. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 0.20
$k_{95.45} = 1.57$

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COMBINATION OF A RECTANGULAR AND A U-SHAPED DISTRIBUTIONS – SIMULATED

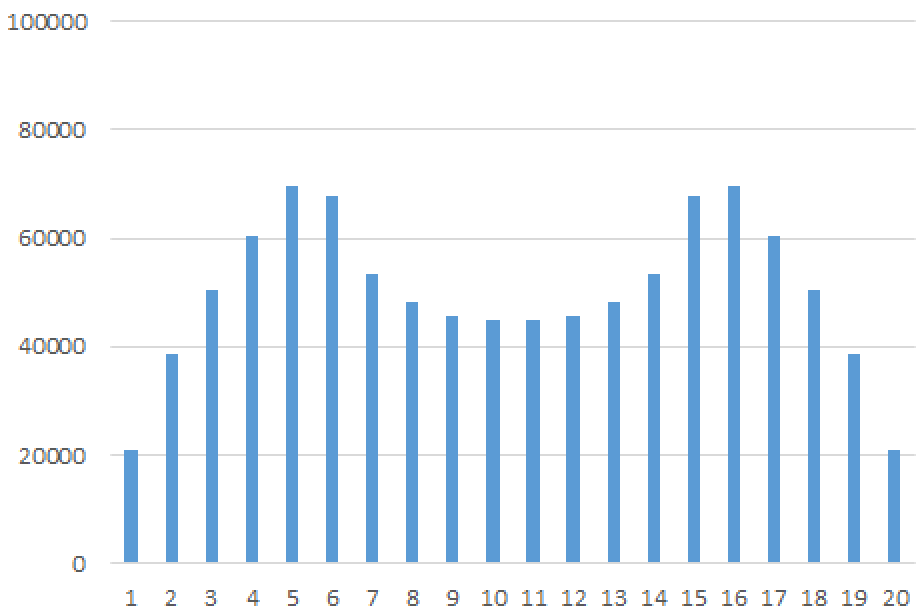

Fig. 35. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 0.30
$k_{95.45}$ = 1.66

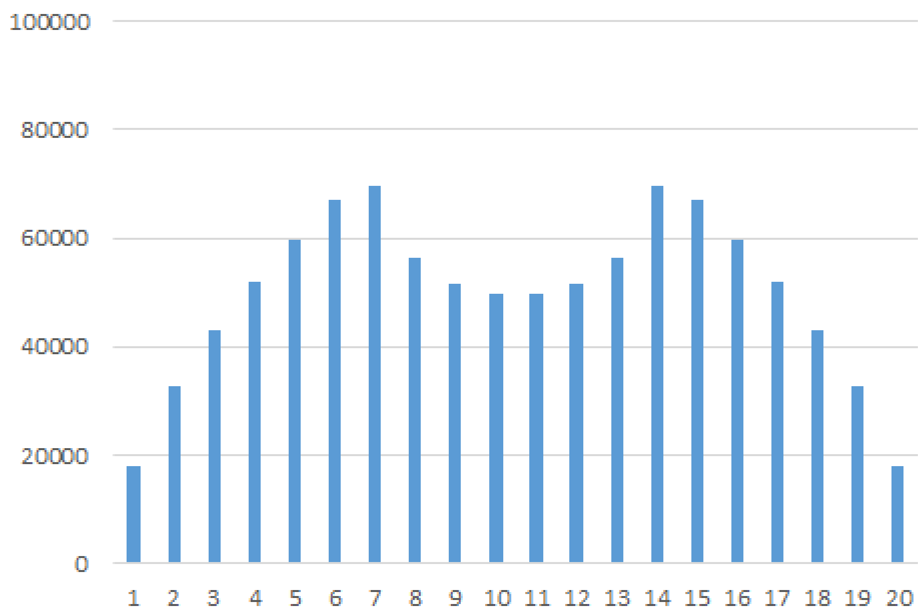

Fig. 36. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 0.40
$k_{95.45}$ = 1.73

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COMBINATION OF A RECTANGULAR AND A U-SHAPED DISTRIBUTIONS – SIMULATED

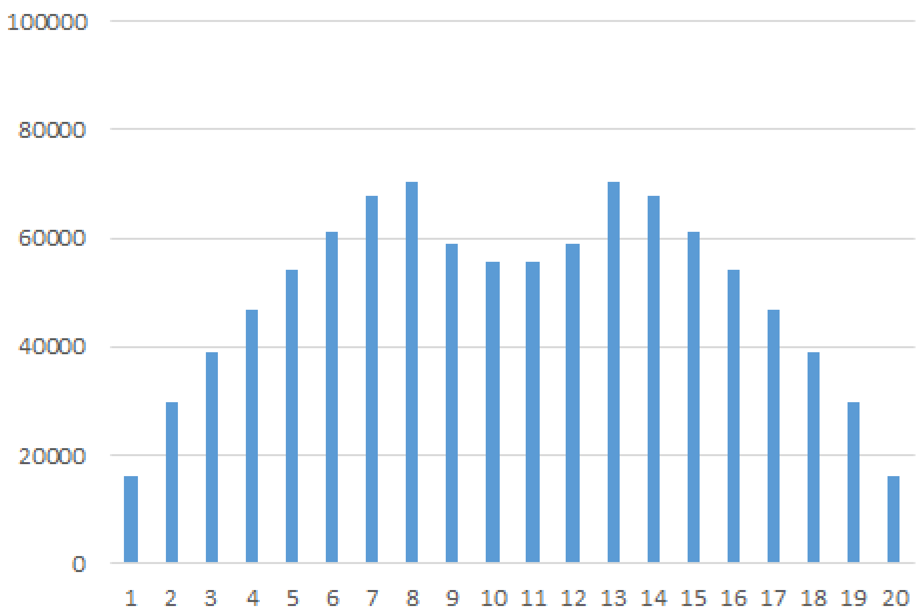

Fig. 37. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 0.50
$k_{95.45}$ = 1.78

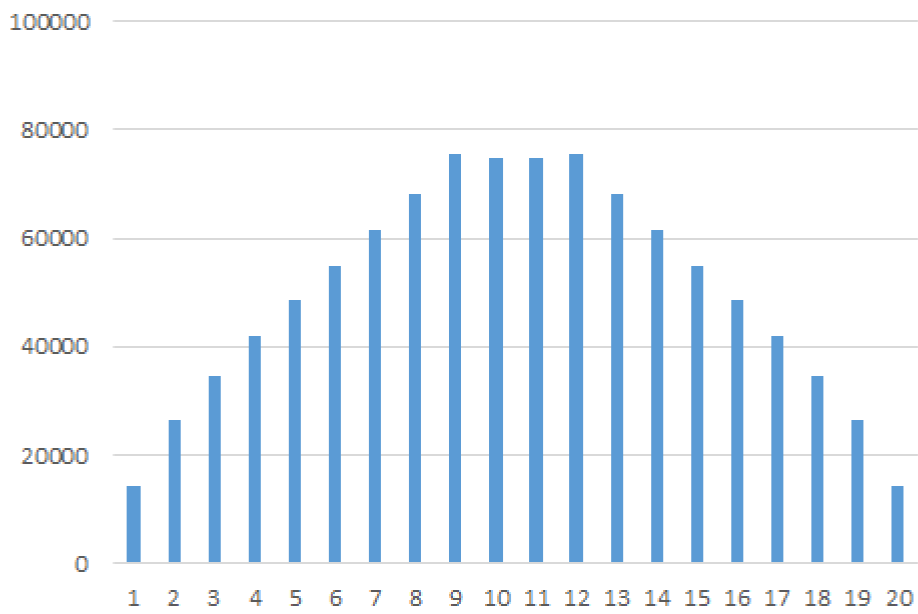

Fig. 38. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 0.70
$k_{95.45}$ = 1.86

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COMBINATION OF A RECTANGULAR AND A U-SHAPED DISTRIBUTIONS – SIMULATED

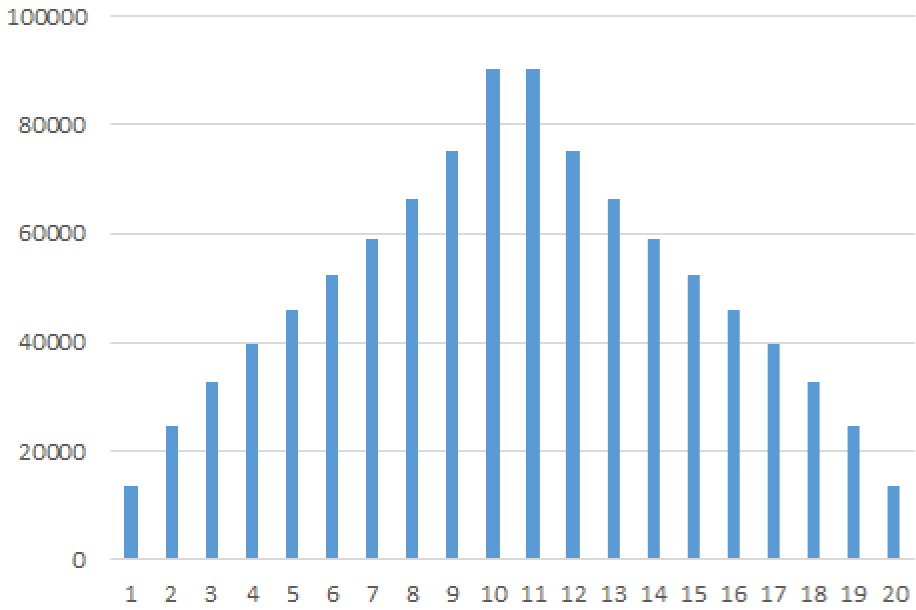

Fig. 39. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 0.90
$k_{95.45}$ = 1.89

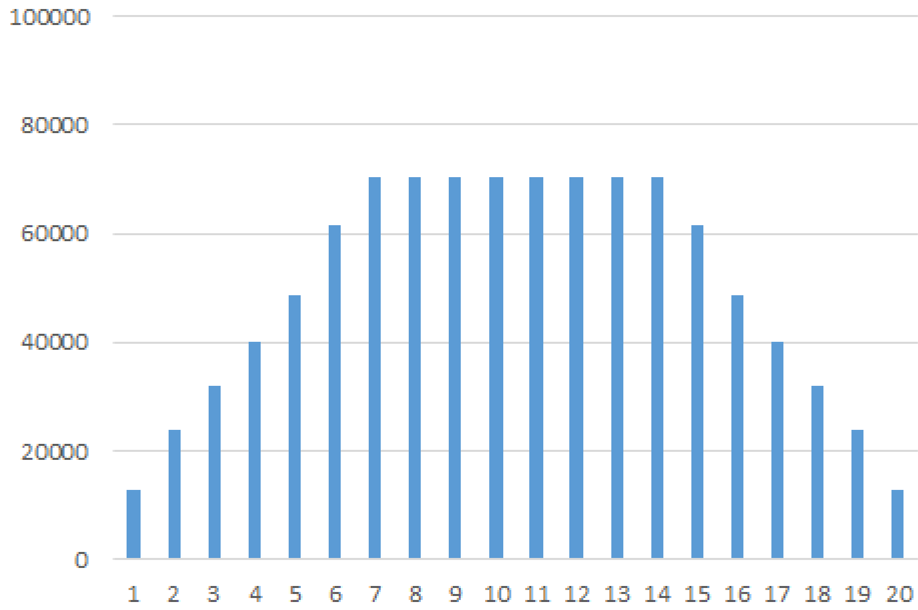

Fig. 40. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 2.00
$k_{95.45}$ = 1.86

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COMBINATION OF A RECTANGULAR AND A U-SHAPED DISTRIBUTIONS – SIMULATED

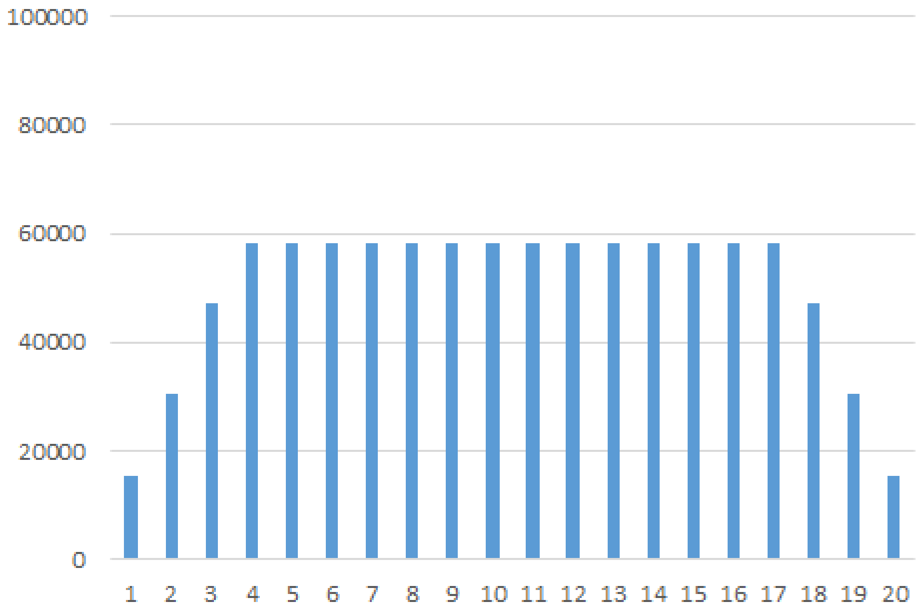

Fig. 41. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 5.00
$k_{95.45}$ = 1.72

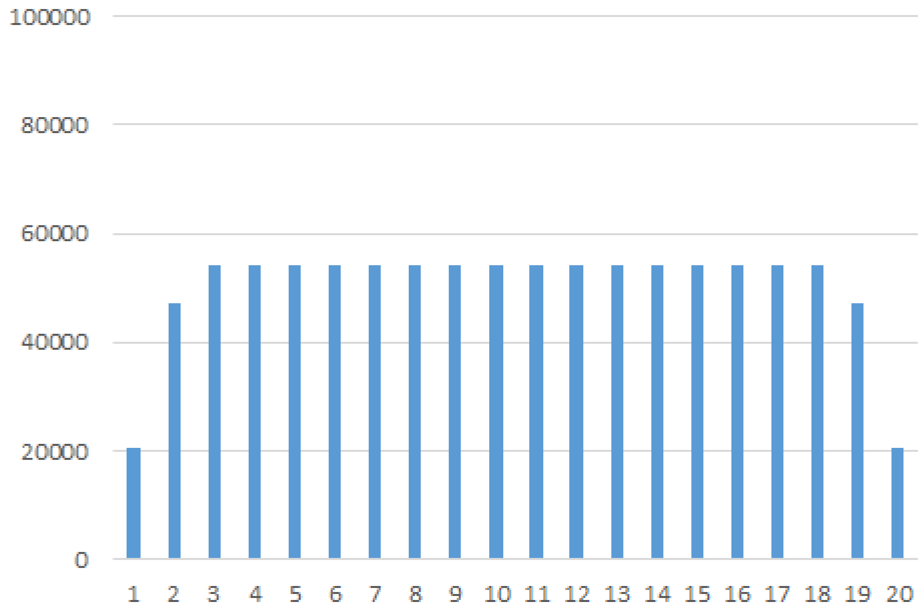

Fig. 42. $u_i(y)$ rectangular / $u_i(y)$ U-shaped = 10.00
$k_{95.45}$ = 1.66

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COMBINATION OF A TWO RECTANGULAR OR TWO U-SHAPED DISTRIBUTIONS – THEORETICAL

Ratio $\frac{u_i(y)_{smaller}}{u_i(y)_{larger}}$	k for stated ratio 2 Rectangular Distributions	k for stated ratio 2 U-shaped Distributions
0.00	1.65	1.41
0.05	1.65	1.44
0.10	1.66	1.49
0.15	1.69	1.53
0.20	1.71	1.58
0.25	1.74	1.62
0.30	1.77	1.66
0.35	1.79	1.69
0.40	1.82	1.72
0.45	1.84	1.75
0.50	1.86	1.77
0.60	1.89	1.81
0.70	1.91	1.83
0.80	1.92	1.85
0.90	1.93	1.86
1.00	1.93	1.86

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COMBINATION OF A TWO RECTANGULAR DISTRIBUTIONS – SIMULATED

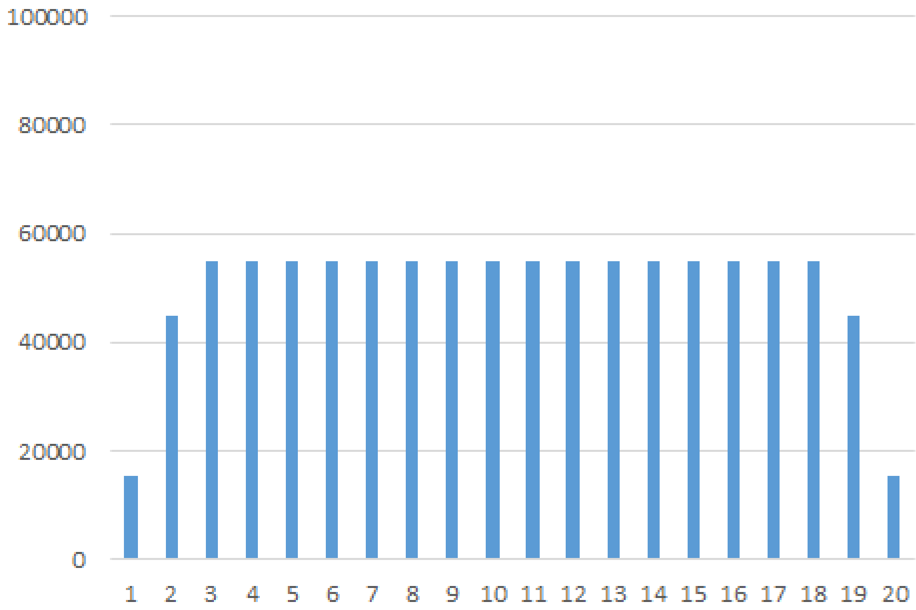

Fig. 43. $u_i(y)$ smaller / $u_i(y)$ larger = 0.10
$k_{95.45}$ = 1.66

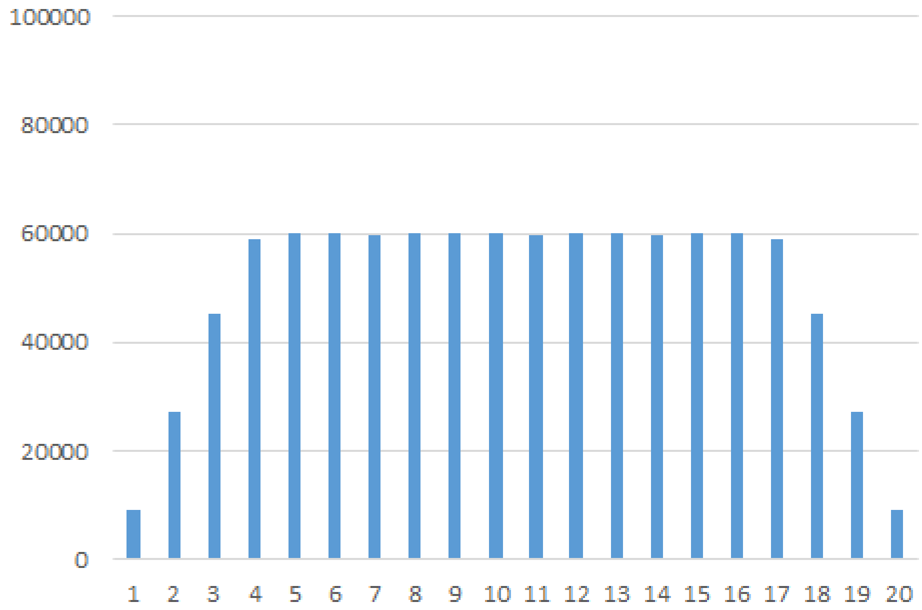

Fig. 44. $u_i(y)$ smaller / $u_i(y)$ larger = 0.20
$k_{95.45}$ = 1.71

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COMBINATION OF TWO RECTANGULAR DISTRIBUTIONS – SIMULATED

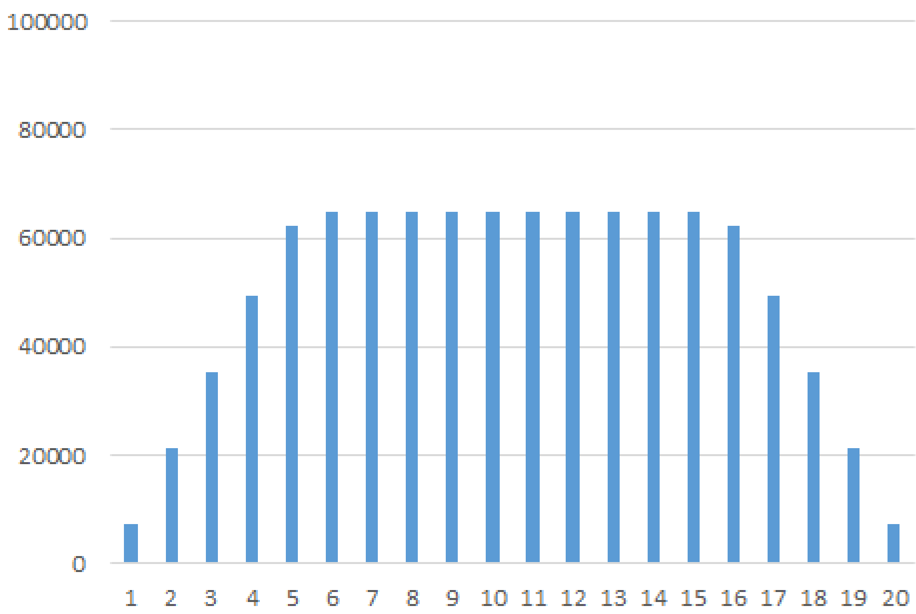

Fig. 45. $u_i(y)$ smaller / $u_i(y)$ larger = 0.30
$k_{95.45}$ = 1.77

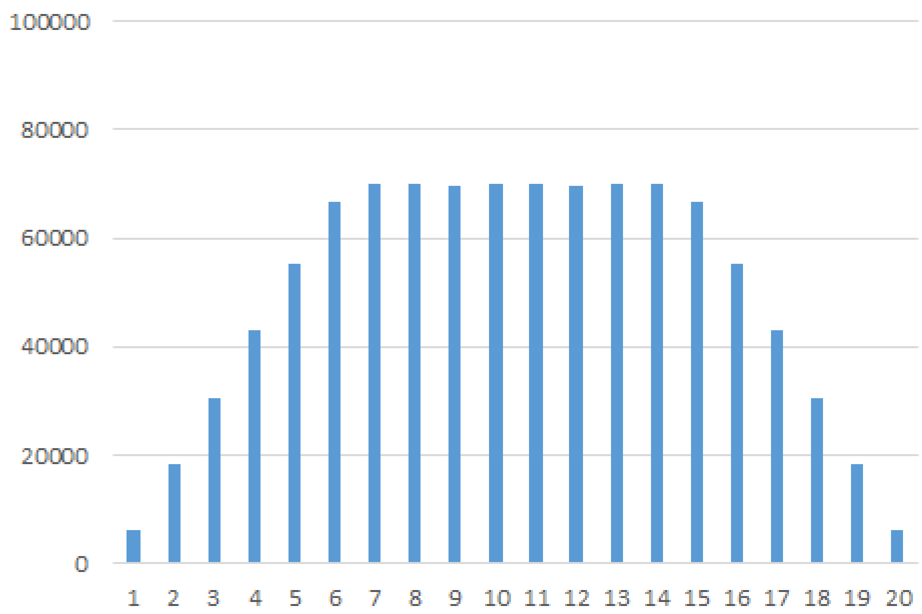

Fig. 46. $u_i(y)$ smaller / $u_i(y)$ larger = 0.40
$k_{95.45}$ = 1.82

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COMBINATION OF TWO RECTANGULAR DISTRIBUTIONS – SIMULATED

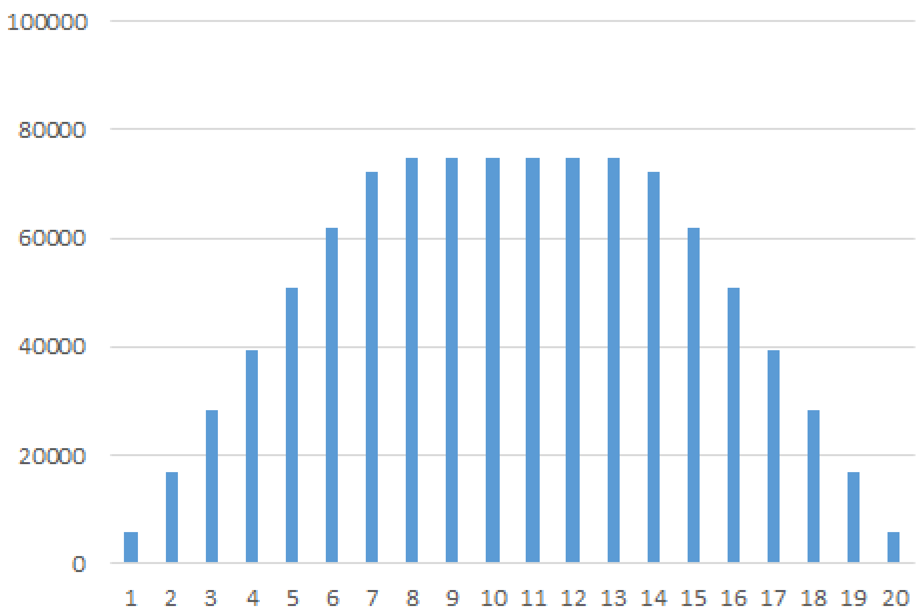

Fig. 47. $u_i(y)$ smaller / $u_i(y)$ larger = 0.50
$k_{95.45}$ = 1.86

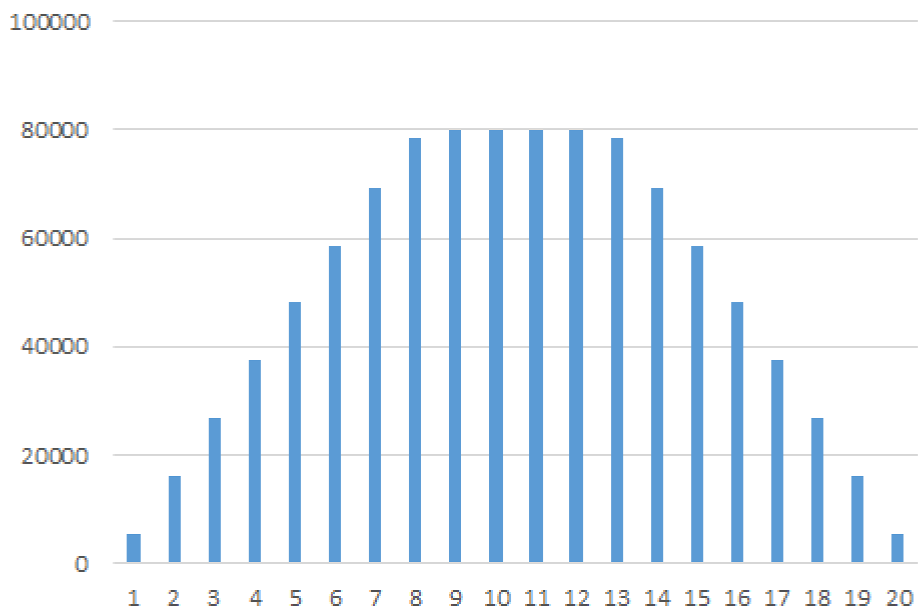

Fig. 48. $u_i(y)$ smaller / $u_i(y)$ larger = 0.60
$k_{95.45}$ = 1.89

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COMBINATION OF TWO RECTANGULAR DISTRIBUTIONS – SIMULATED

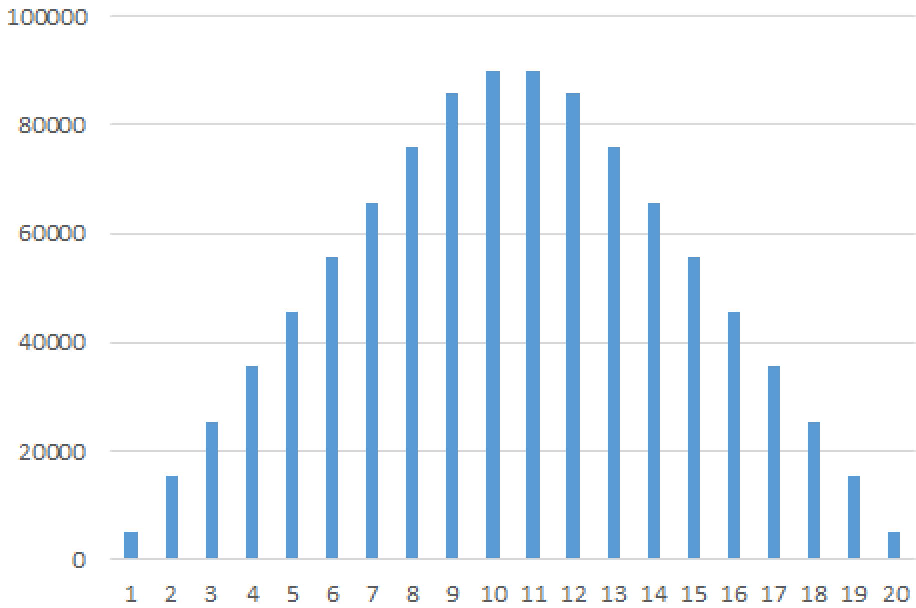

Fig. 49. $u_i(y)$ smaller / $u_i(y)$ larger = 0.80
$k_{95.45}$ = 1.92

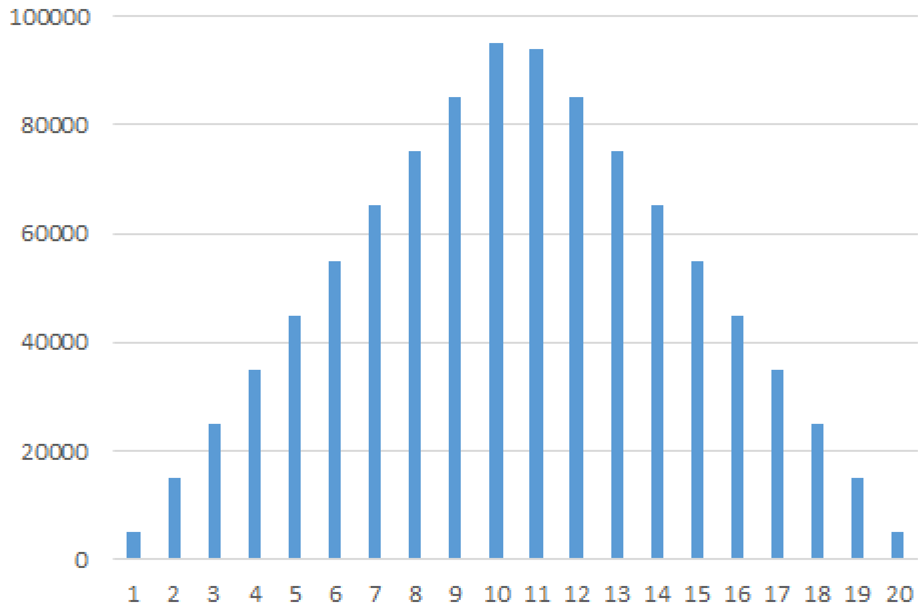

Fig. 50. $u_i(y)$ smaller / $u_i(y)$ larger = 1.00
$k_{95.45}$ = 1.93

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COMBINATION OF A TWO U-SHAPED DISTRIBUTIONS – SIMULATED

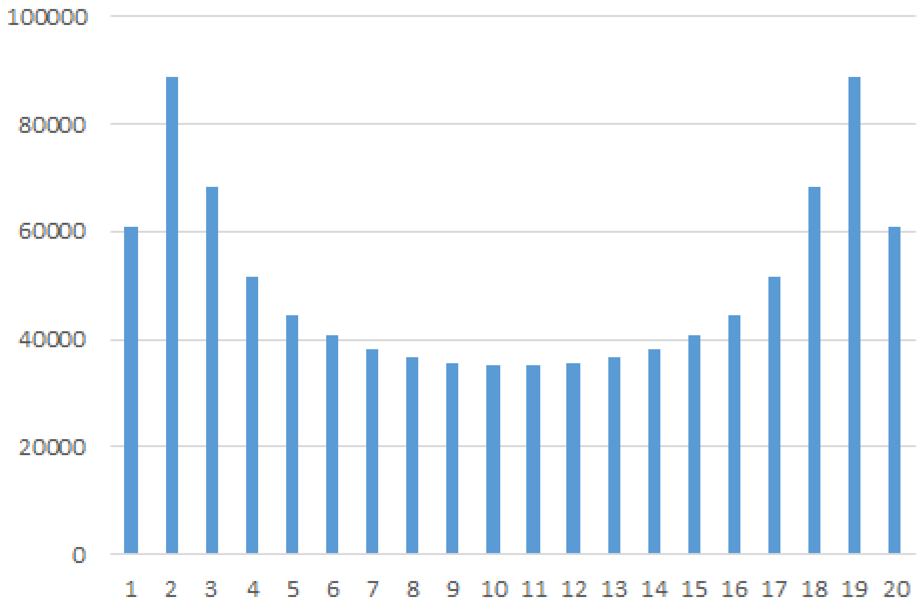

Fig. 51. $u_i(y)$ smaller / $u_i(y)$ larger = 0.10
$k_{95.45} = 1.49$

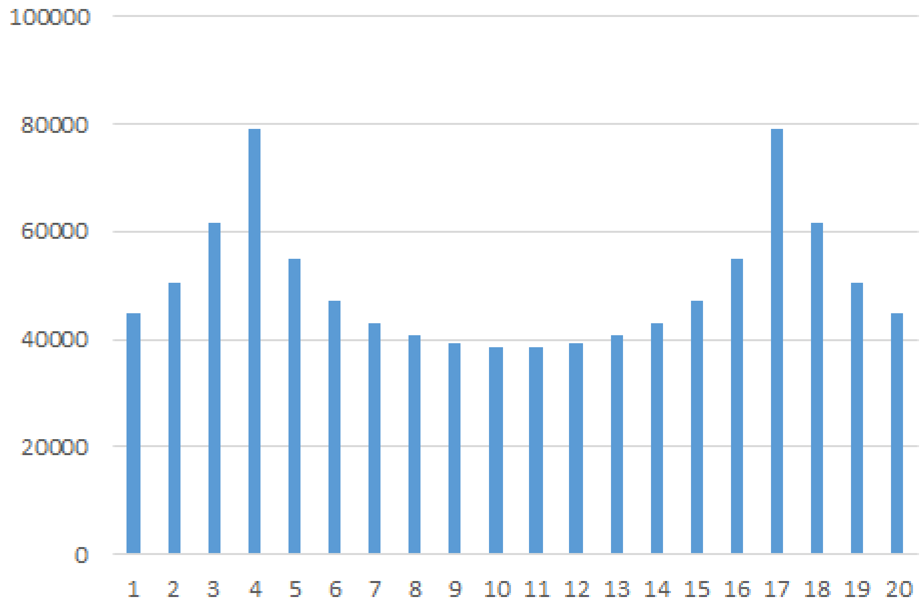

Fig. 52. $u_i(y)$ smaller / $u_i(y)$ larger = 0.20
$k_{95.45} = 1.58$

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COMBINATION OF TWO U-SHAPED DISTRIBUTIONS – SIMULATED

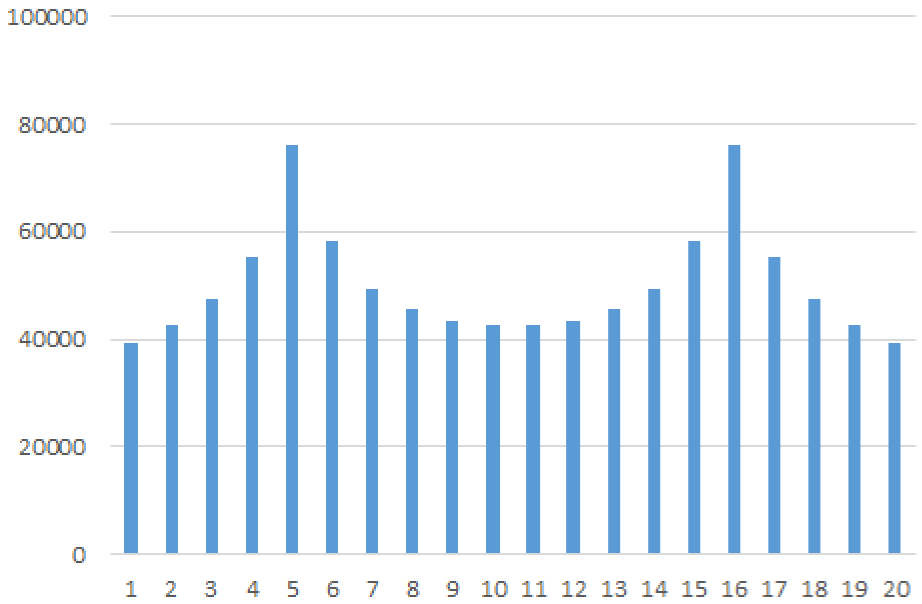

Fig. 53. $u_i(y)$ smaller / $u_i(y)$ larger = 0.30
$k_{95.45}$ = 1.66

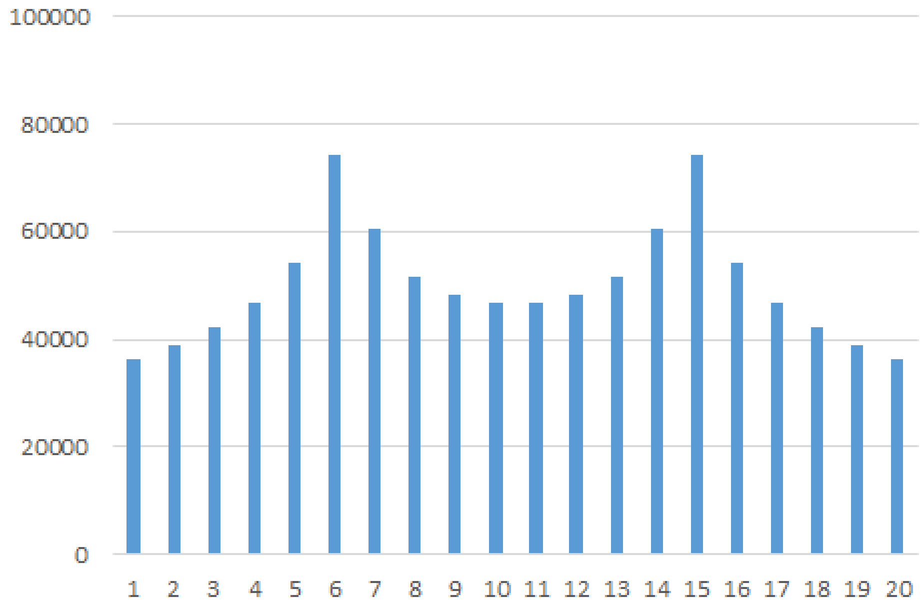

Fig. 54. $u_i(y)$ smaller / $u_i(y)$ larger = 0.40
$k_{95.45}$ = 1.72

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COMBINATION OF TWO U-SHAPED DISTRIBUTIONS – SIMULATED

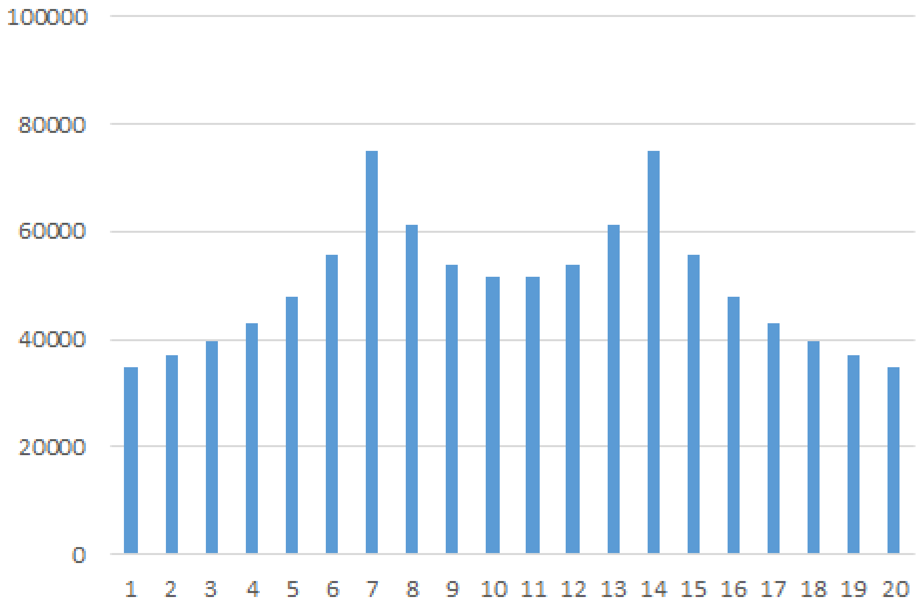

Fig. 55. $u_i(y)$ smaller / $u_i(y)$ larger = 0.50
$k_{95.45}$ = 1.77

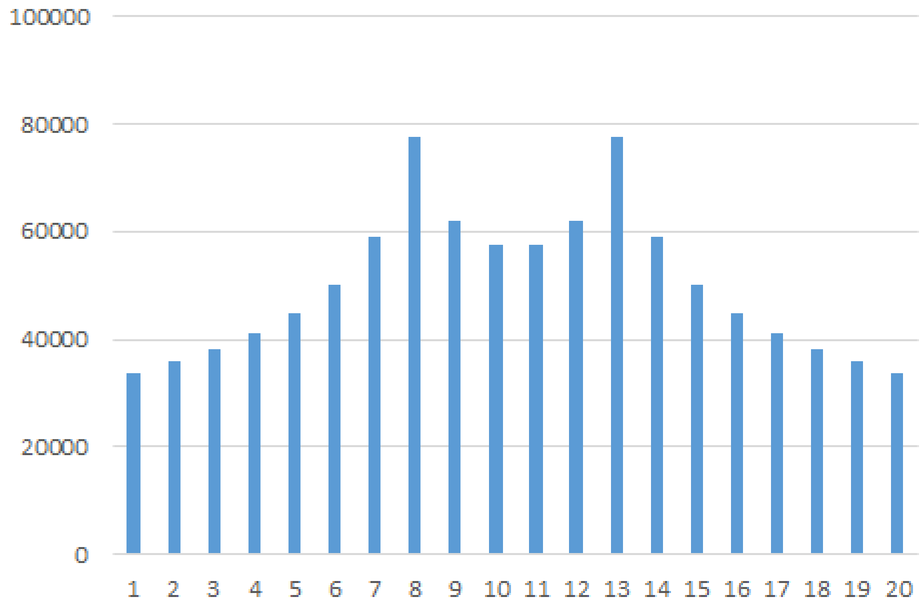

Fig. 56. $u_i(y)$ smaller / $u_i(y)$ larger = 0.60
$k_{95.45}$ = 1.81

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COMBINATION OF TWO U-SHAPED DISTRIBUTIONS – SIMULATED

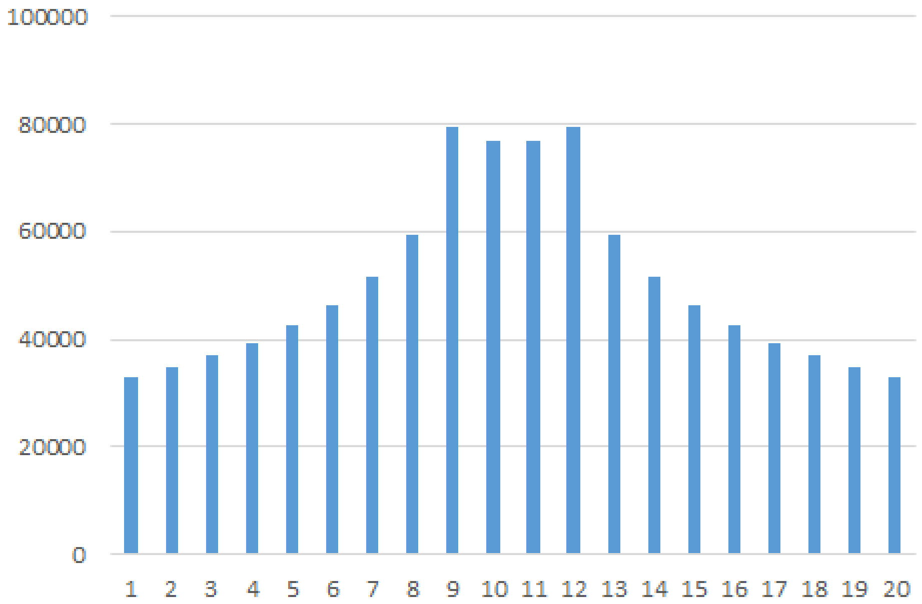

Fig. 57. $u_i(y)$ smaller / $u_i(y)$ larger = 0.80
$k_{95.45} = 1.85$

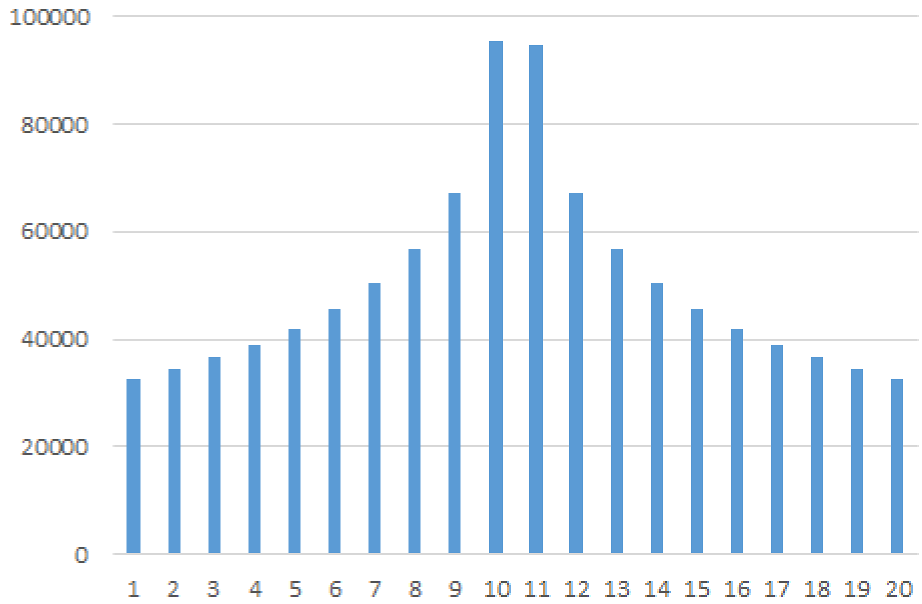

Fig. 58. $u_i(y)$ smaller / $u_i(y)$ larger = 1.00
$k_{95.45} = 1.86$

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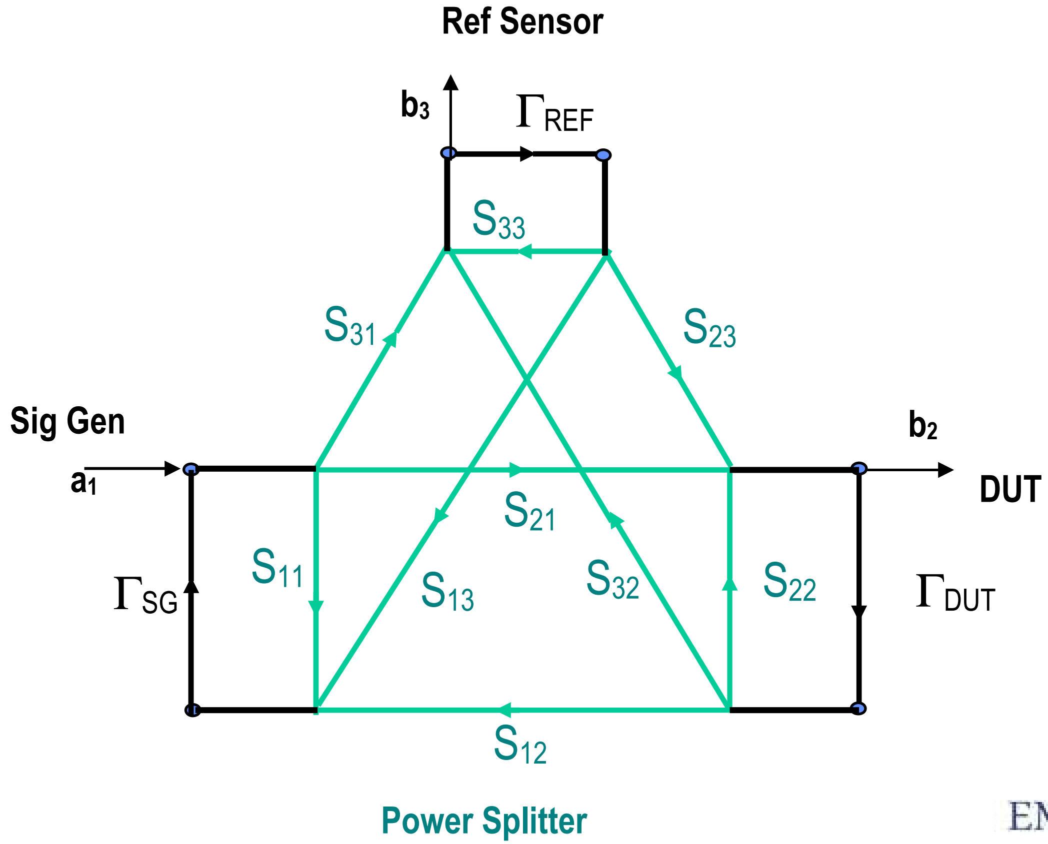
APPLICATION TO THE CHARACTERISATION OF POWER SENSORS

The Direct 3-port Method for mesasurement of Calibration Factor (after Ken Wong, Keysight Technoloties): a standard approach to measure CF is to compare the Calibration Factor of a power sensor under test to that of a reference sensor with known Calibration Factor. Typically, a power splitter is used to split the incident signal into two equal parts. The reference sensor is connected to one of the output ports of the splitter and the DUT sensor is connected to the other output port. This is illustrated in the figure below.



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THE EXPRESSION OF THE MEASURED CALIBRATION FACTOR – INCLUDING MISMATCH CORRECTION

The expression of the measured Calibration Factor of the sensor under test is as follows. It contains the four readings taken from the three power sensors involved, the known Calibration Factor of the Standard sensor (which assures traceability of the method) and the Mismatch correction.

Usually the Mismatch correction is ignored since we are lacking some of the information required (the three reflection coefficients involved must be known in magnitude and phase). But in this particular case both the splitter and the power sensors can be extracted from the setup and measured independently. The Mismatch correction, as it can be seen, is a scalar correction dependent on complex quantities, which makes it difficult to derivate:

$$CF_{DUT} = CF_{Std} \cdot \left(\frac{M_{DUT}}{M_{Ref(DUT)}} \right) \cdot \left(\frac{M_{Ref(Std)}}{M_{Std}} \right) \cdot \left| \frac{1 - \Gamma_{DUT}\Gamma_{Eq2}}{1 - \Gamma_{Std}\Gamma_{Eq2}} \right|^2$$

- ✓ CF_{DUT} is the Calibration Factor of the power sensor under test, and Γ_{DUT} its complex reflection coefficient
- ✓ CF_{Std} is the Calibration Factor of the standard power sensor, and Γ_{Std} its complex reflection coefficient
- ✓ M_{DUT} is the reading with the power sensor under test
- ✓ M_{Std} is the reading with the standard power sensor
- ✓ $M_{Ref(DUT)}$ is the reading with the reference power sensor at Port 3 when the power sensor under test is attached to Port 2 of the splitter
- ✓ $M_{Ref(Std)}$ is the reading with the reference power sensor at Port 3 when the standard power sensor is attached to Port 2 of the splitter
- ✓ Γ_{Eq2} is the complex equivalent Source Match of the power splitter at Port 2

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THE MISMATCH CORRECTION AND ITS ASSOCIATED UNCERTAINTY

The following expression can be derived for the uncertainty associated to the Mismacth correction. It is based on simple derivatives with the assumption that the three independent variables involved are scalar.

$$U_M = 100 \cdot \left(\frac{2U_{\Gamma(DUT)}\left|\Gamma_{Eq2}\right| + 2U_{\Gamma(Eq2)}\left|\Gamma_{DUT}\right|}{\left|1-\Gamma_{DUT}\Gamma_{Eq2}\right|} + \frac{2U_{\Gamma(Std)}\left|\Gamma_{Eq2}\right| + 2U_{\Gamma(Eq2)}\left|\Gamma_{Std}\right|}{\left|1-\Gamma_{Std}\Gamma_{Eq2}\right|} \right) \quad \textit{Worst case}$$

$$U_M = 100 \cdot \sqrt{\frac{4U^2_{\Gamma(DUT)}\left|\Gamma_{Eq2}\right|^2 + 4U^2_{\Gamma(Eq2)}\left|\Gamma_{DUT}\right|^2}{\left|1-\Gamma_{DUT}\Gamma_{Eq2}\right|^2} + \frac{4U^2_{\Gamma(Std)}\left|\Gamma_{Eq2}\right|^2 + 4U^2_{\Gamma(Eq2)}\left|\Gamma_{Std}\right|^2}{\left|1-\Gamma_{Std}\Gamma_{Eq2}\right|^2}} \quad \textit{Root sum of squares}$$

It is however difficult to compute the corresponding expression when three complex quantities are present (the three reflection coefficients in magnitude and phase). It is time therefore to make use of Monte Carlo in order to estimate the uncertainty associated to the Mismatch correction.

The case study is the characterisation of a power sensor at 23 GHz with the following complex reflection coefficients involved (nominal Mismatch correction is 1.0175):

$$\Gamma_{DUT} = 0.141\big|_{\angle -99.3^\circ} \qquad \Gamma_{Std} = 0.034\big|_{\angle 31.4^\circ} \qquad \Gamma_{Eq2} = 0.053\big|_{\angle -68.3^\circ}$$

$$U_{\Gamma(DUT)} = 0.014 \qquad U_{\Gamma(Std)} = 0.020 \qquad U_{\Gamma(Eq2)} = 0.011$$

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A SIMPLE APPROACH TO MONTE CARLO: ESTIMATION OF BOUNDS

- ✓ In order to estimate bounds for the above expression, let us sum up onto each nominal value of complex reflection coefficient the five (5) following vectors: (0, 0), (0, U_Γ), (0, - U_Γ), (U_Γ,0) and (-U_Γ, 0). This gives us five 'offset' positions for each reflection coefficient.

- ✓ As a second step, let us write down the 5 times x 5 times x 5 times = 125 possible combinations of the five 'offset' positions of the three reflection coefficients, and compute the Mismatch correction (in magnitude and phase).

- ✓ Finally, let us represent the constellation of complex points obtained and the histogram of the Mismatch correction (this being a scalar correction, we just retain the magnitude of the above constellation of points).

- ✓ Note that we have simulated the contribution of each complex reflection coefficient as a rough contour with four (4) points plus the mean value.

- ✓ With the scalar formula for determination of the Mismatch uncertainty the assumed shape of the probability density function is unknown. With this simplified Monte Carlo a first look at the 'roughly computed' pdf may suggest a triangular distribution. At any case, we would not recommend the rectangular nor the U-shaped distribution for the mismatch uncertainty.

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A SIMPLE APPROACH TO MONTE CARLO: ESTIMATION OF BOUNDS

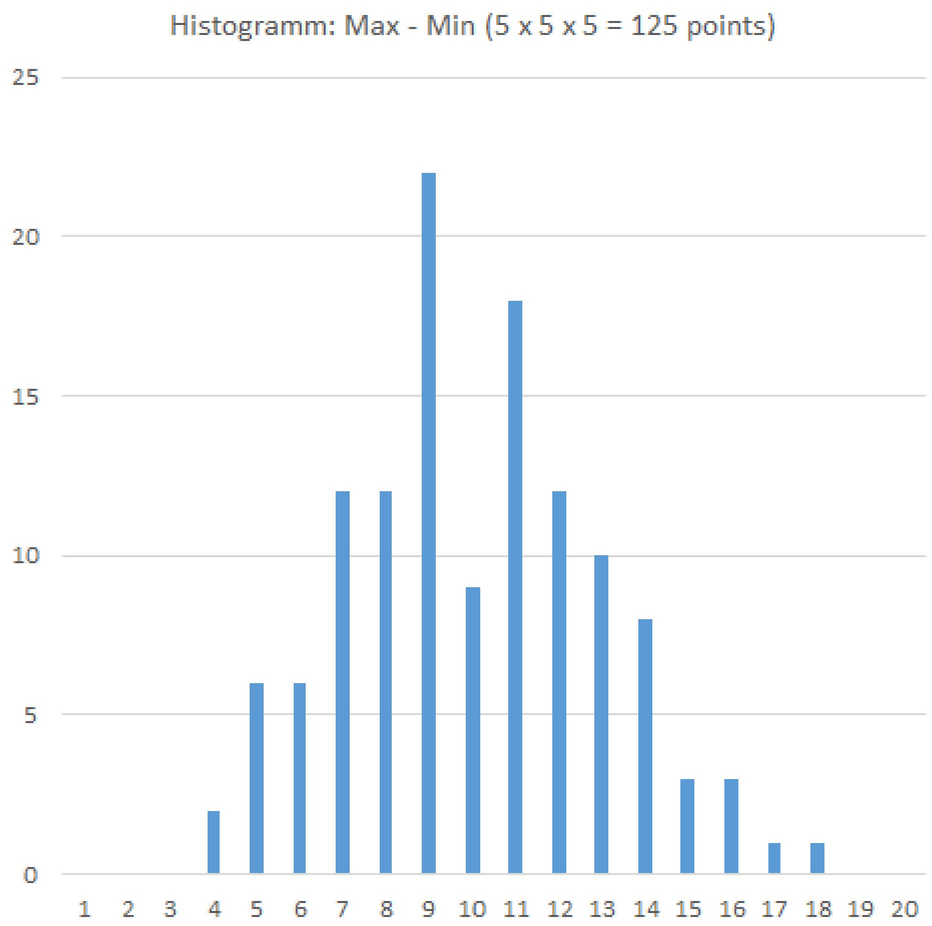

Fig. 59. Histogram between 1.008 and 1.028

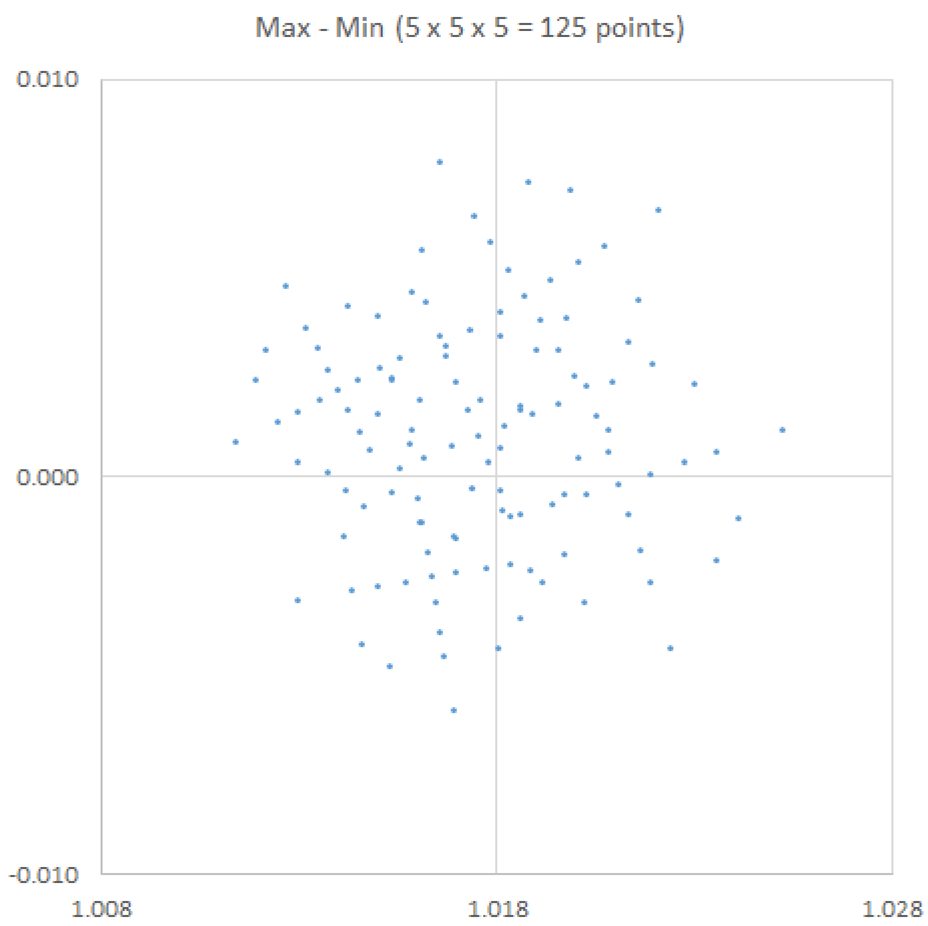

Fig. 60. Complex mismatch term (constellation)

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A SIMPLE APPROACH TO MONTE CARLO: FINER ESTIMATION OF BOUNDS

- ✓ Let us sum up onto each nominal value of complex reflection coefficient the nine (9) following vectors: (0, 0), (0, U_Γ), (0, - U_Γ), (U_Γ,0), (-U_Γ, 0), ($U_\Gamma/\sqrt{2}$, $U_\Gamma/\sqrt{2}$), (-$U_\Gamma/\sqrt{2}$, -$U_\Gamma/\sqrt{2}$), ($U_\Gamma/\sqrt{2}$, -$U_\Gamma/\sqrt{2}$) and (-$U_\Gamma/\sqrt{2}$, $U_\Gamma/\sqrt{2}$) This gives us nine 'offset' positions for each reflection coefficient.

- ✓ As a second step, let us write down the 9 times x 9 times x 9 times = 729 possible combinations of the nine 'offset' positions of the three reflection coefficients, and compute the Mismatch correction (in magnitude and phase).

- ✓ Finally, let us represent the constellation of complex points obtained and the histogram of the Mismatch correction.

- ✓ We have simulated the contribution of each complex reflection coefficient as a finer contour with eight (8) points plus the mean value.

- ✓ NOTE: we could think of further refinements of the method with thirteen (13) vectors: (0, 0), (0, U_Γ), (0, - U_Γ), (U_Γ,0), (-U_Γ, 0), ($\sqrt{3}U_\Gamma/2$, $U_\Gamma/2$), ($U_\Gamma/2$, $\sqrt{3}U_\Gamma/2$), (-$\sqrt{3}U_\Gamma/2$, -$U_\Gamma/2$), (-$U_\Gamma/2$, -$\sqrt{3}U_\Gamma/2$), ($\sqrt{3}U_\Gamma/2$, -$U_\Gamma/2$), ($U_\Gamma/2$, -$\sqrt{3}U_\Gamma/2$), (-$\sqrt{3}U_\Gamma/2$, $U_\Gamma/2$) and (-$U_\Gamma/2$, $\sqrt{3}U_\Gamma/2$). But we would be complicating a method which by definition must be kept simple.

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A SIMPLE APPROACH TO MONTE CARLO: FINER ESTIMATION OF BOUNDS

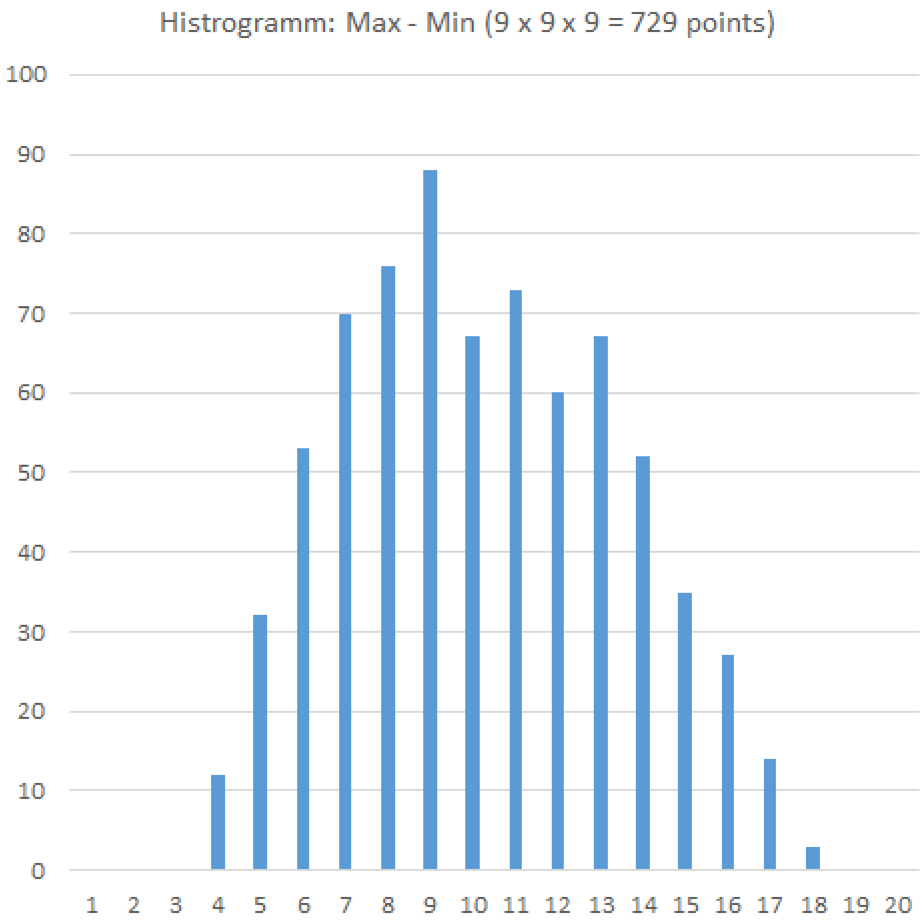

Fig. 61. Histogram between 1.008 and 1.028

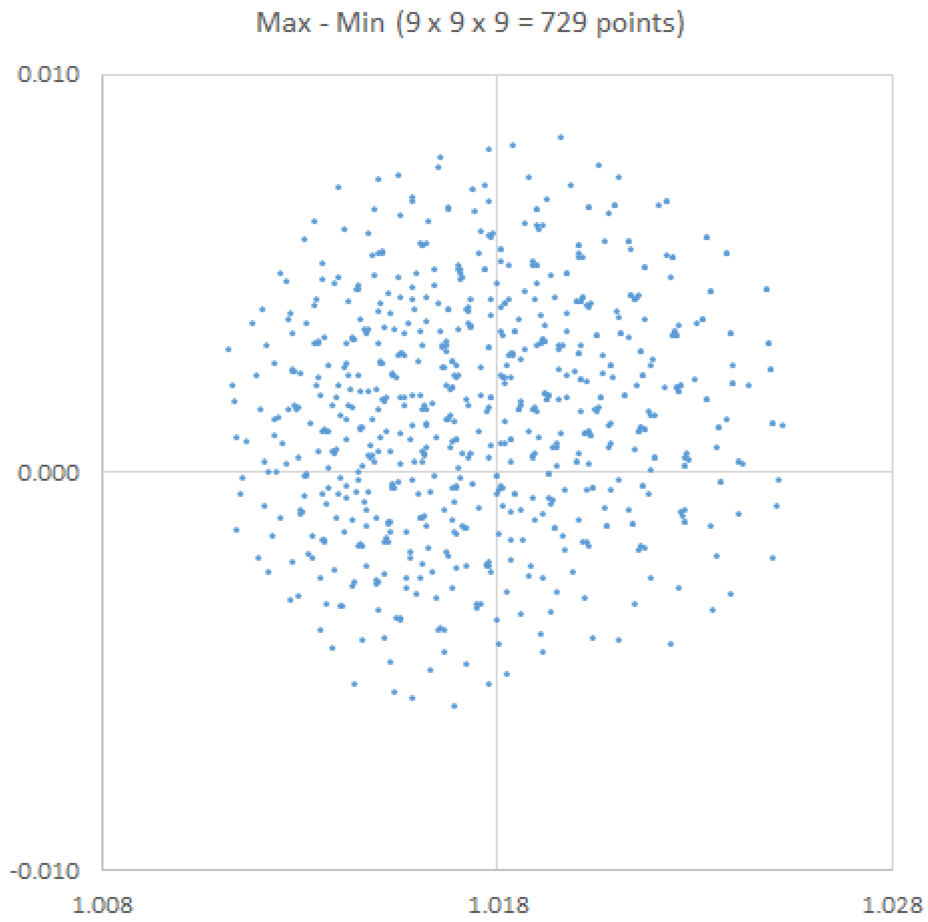

Fig. 62. Complex mismatch term (constellation)

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STRATIFIED SAMPLING WITH 1000 POINTS

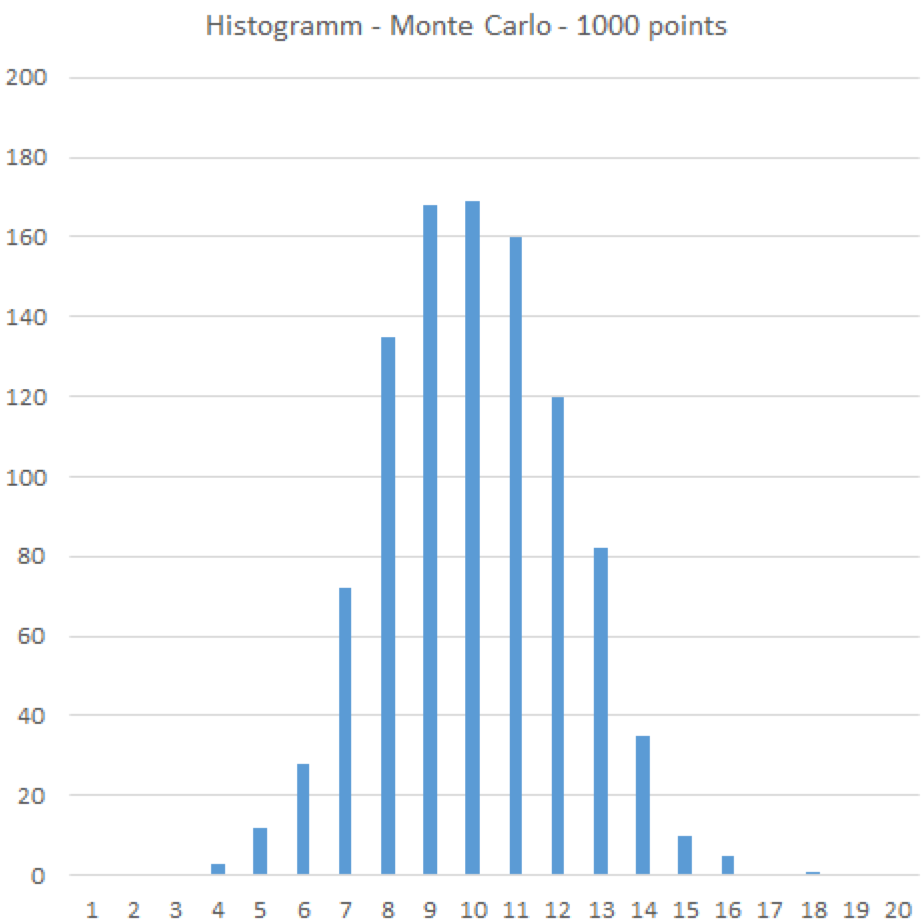

Fig. 63. Histogram between 1.008 and 1.028

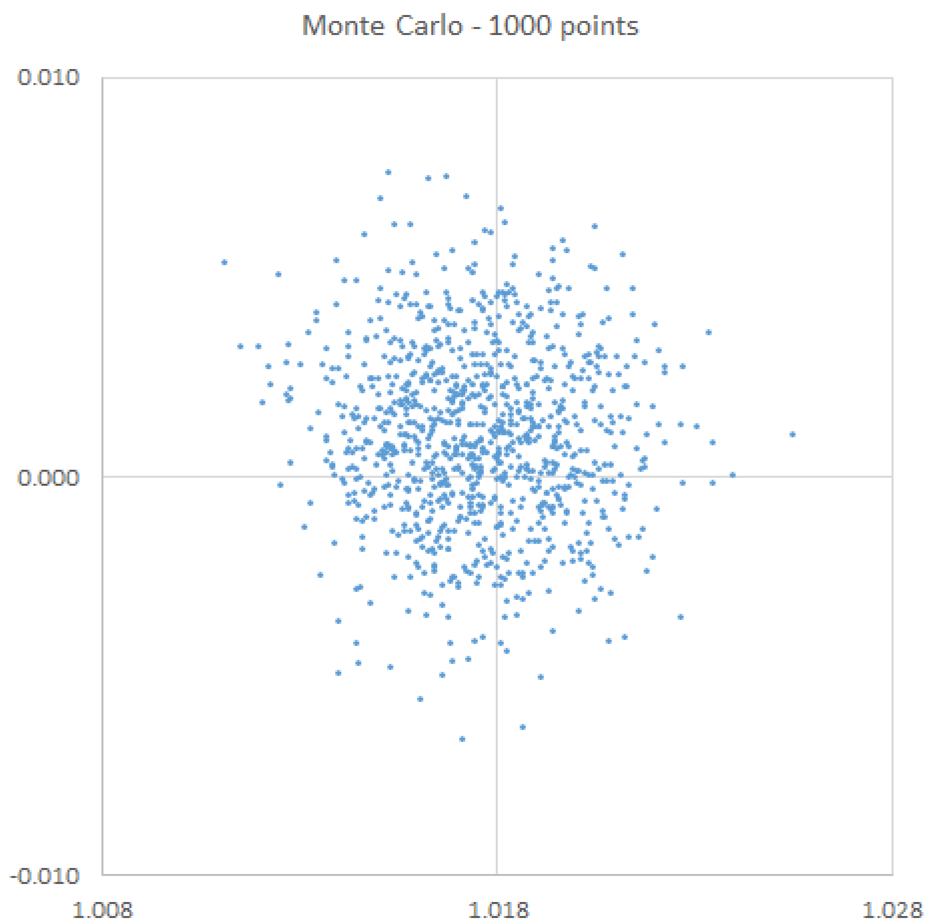

Fig. 64. Complex mismatch term (constellation)

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STRATIFIED SAMPLING WITH 10000 POINTS

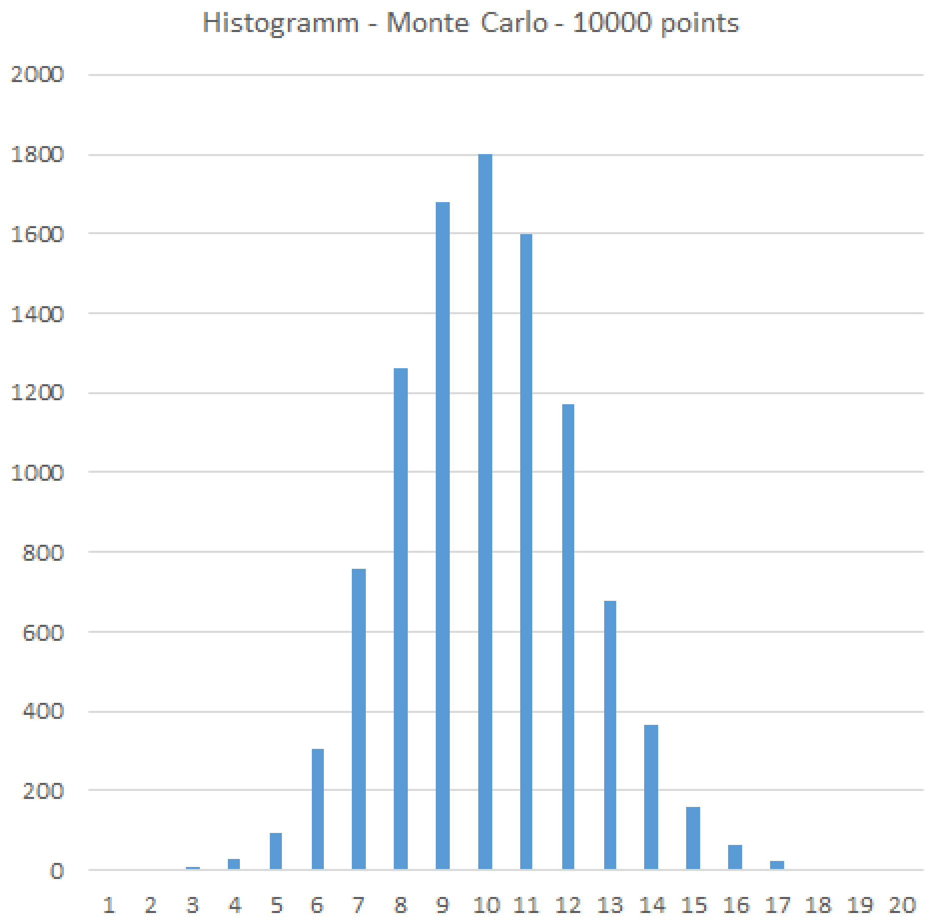

Fig. 65. Histogram between 1.008 and 1.028

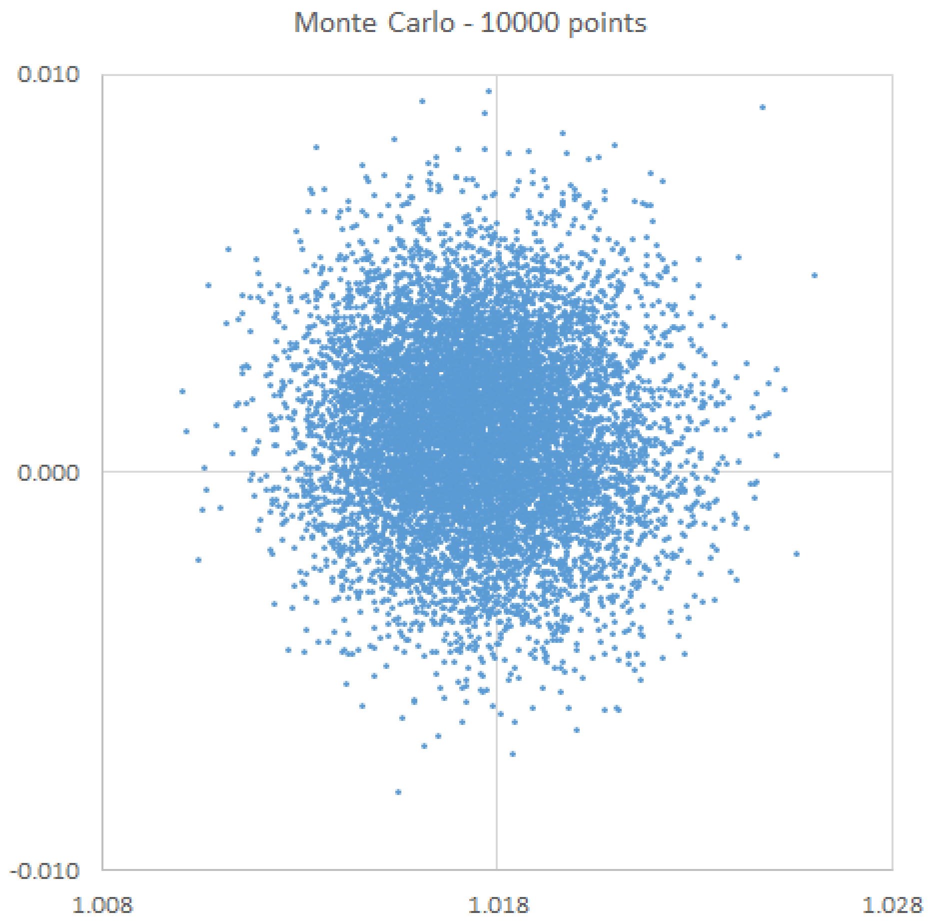

Fig. 66. Complex mismatch term (constellation)

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COMPARISON BETWEEN THE DIFFERENT APPROACHS IN TERMS OF MISMATCH UNCERTAINTY

In the following table the results obtained for the Mismatch uncertainty are compared for the three methods so far presented: the scalar approach, the estimation of bounds with 125 and 729 points and Monte Carlo with Stratified Sampling using 1000 and 10000 points.

	(Max – Min)/(2·√3) (assuming Rectangular)	(Max – Min)/(2·√6) (assuming Triangular)	Sigma
Scalar formula (W.c.)	0.430	-	-
Scalar formula (R.s.s.)	0.312	-	-
Estimation of bounds 125 points	0.393	0.278	0.286
Estimation of bounds 927 points	0.398	0.281	0.316
MC Stratified sampling 1000 points	-	-	0.205 – 0.220
MC Stratified sampling 10000 points	-	-	0.212 – 0.216

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CONCLUSIONS

- ✓ The method of Monte Carlo is based on the random generation of data following a set of desired pdf's. However, not always the whole process has to be pure random. Stratified Sampling helps us define predefined, well-known histograms at the level of precision required by our computations.

- ✓ Key point for Stratified Sampling is the re-arrangement of 'shuffling' of the resulting arrays. Pre-defined functions such as **randperm** in Matlab® or **RANK** in Excel® may help us with this purpose.

- ✓ Occasions in which Monte Carlo shows its utility is when complex expressions are involved which we are not able to (or are not willing to) partially derivate. Mismatch correction is a good example since the formulas found in the literature are often based on real variables.

- ✓ Further possible simplifications of the Monte Carlo method consist in setting bounds to complex expressions. This is adequate in cases where the number of variables is manageable. For example, the combination of three, four, up to five variables encircled within a rough contour with four (4) points may lead to clarifying results.

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